Topology: simplicial methods

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Topological complexes: the idea

Use combinatorics to define the space structure

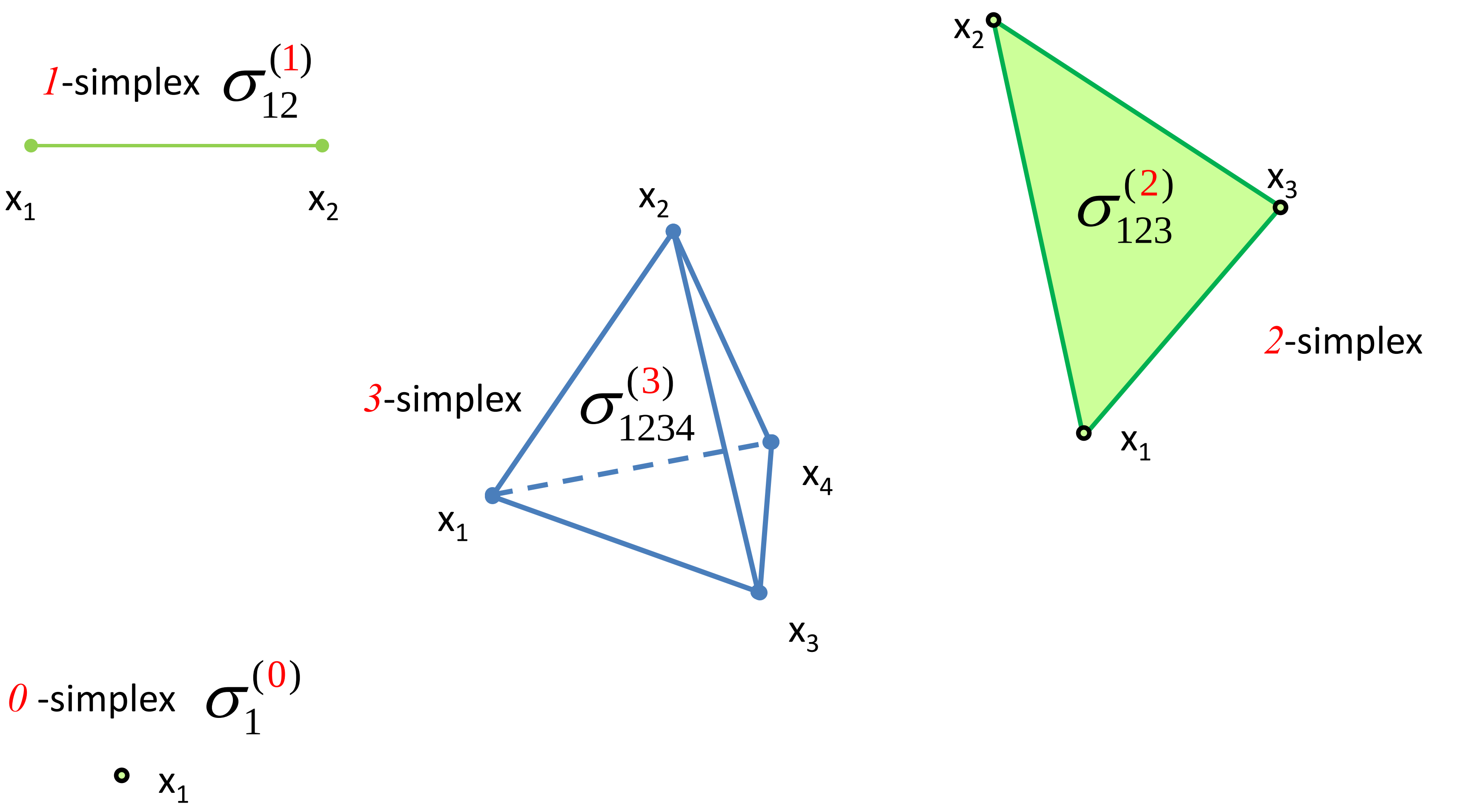
Think of a space as of an “assembly”, a “complex”

- a composition of “elementary pieces”

Use algebraic methods for the analysis

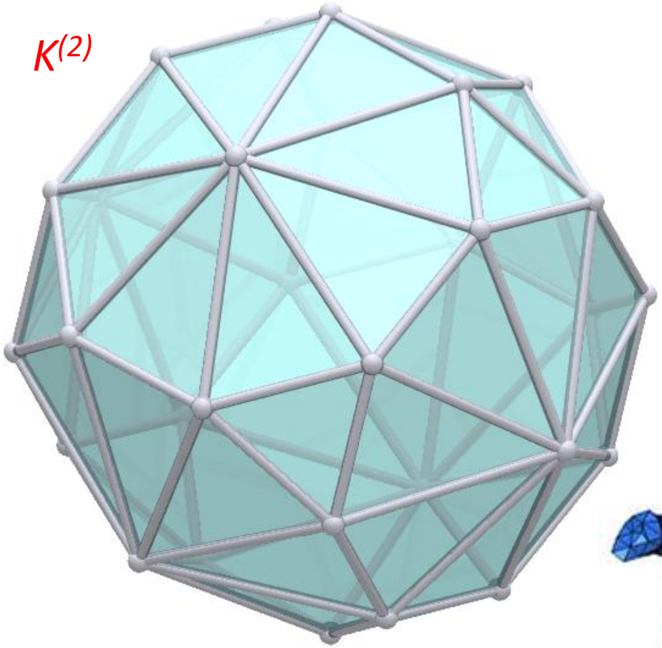
What can serve as an “elementary piece”?

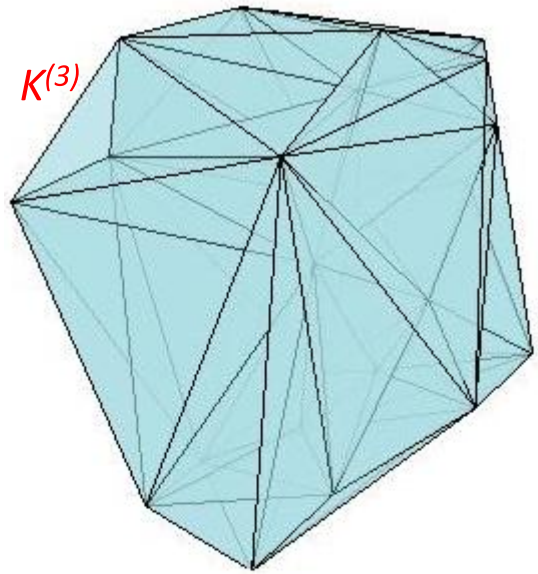
Simplexes

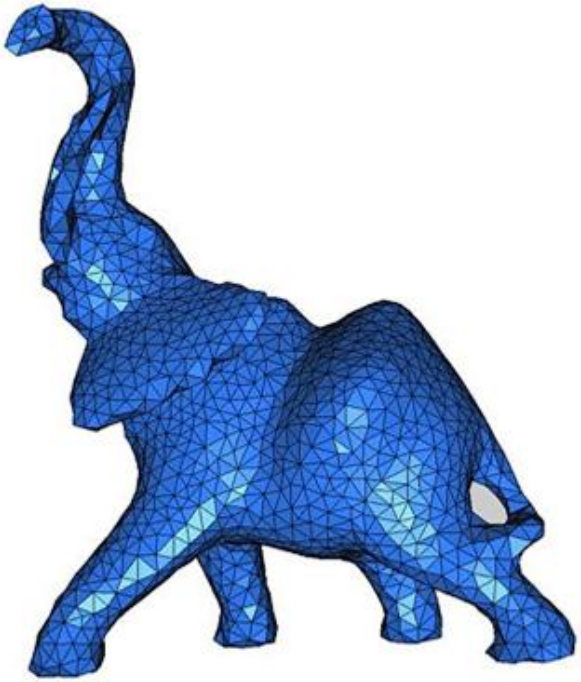


1-simplex $\sigma_{12}^{(1)}$
x_1
x_2
3-simplex $\sigma_{1234}^{(3)}$
x_1
x_2
x_3
x_4
2-simplex $\sigma_{123}^{(2)}$
x_1
x_2
x_3
0 -simplex $\sigma_1^{(0)}$
x_1

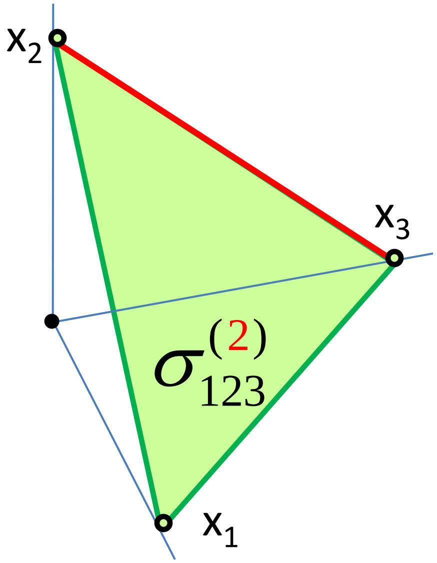
Simplicial complex K

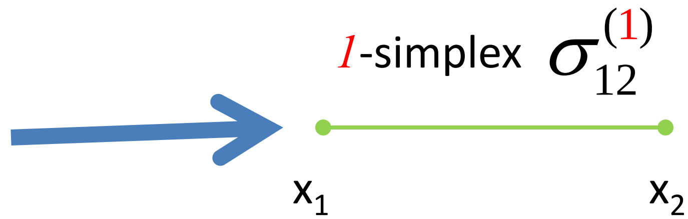


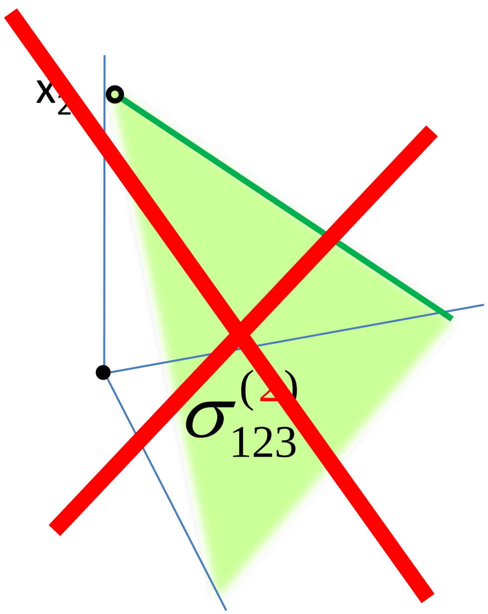




Face of the simplex



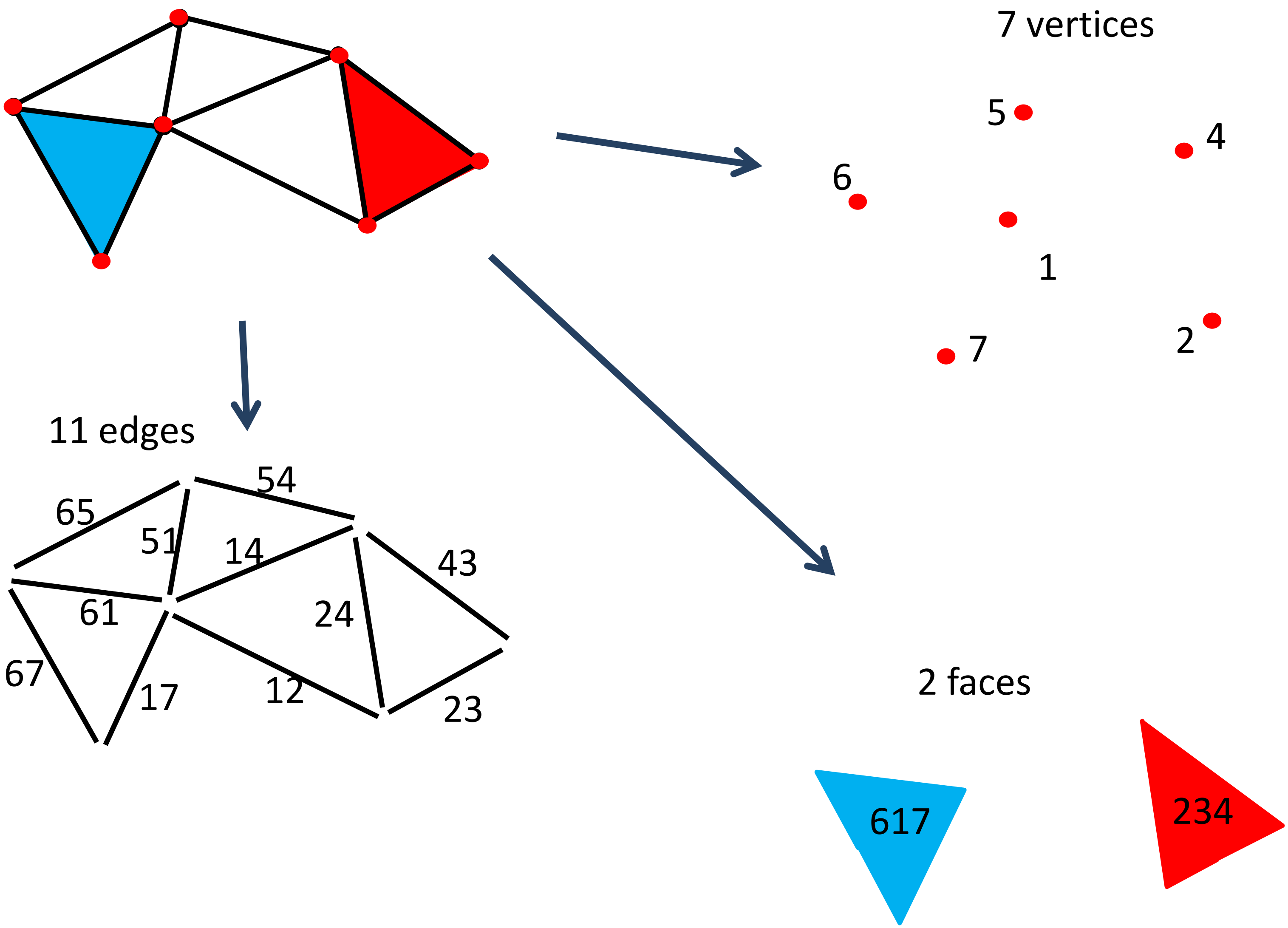




Face of a n-simplex is a $n-1$ simplex

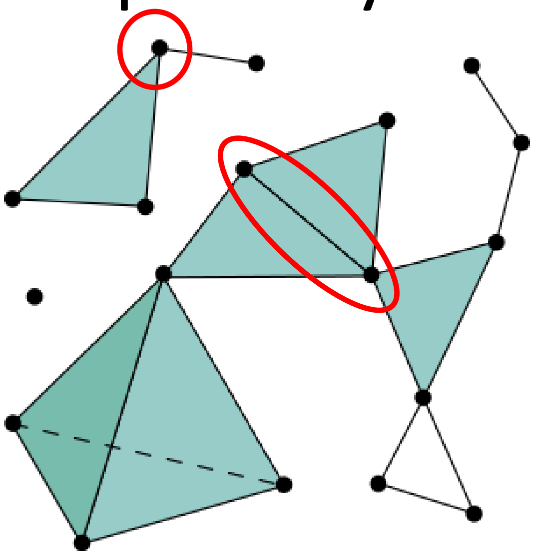
Simplex contains all of its faces

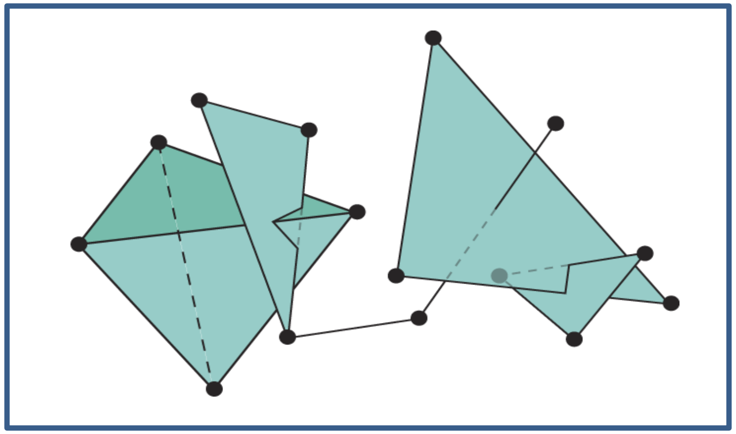
Simplicial complex *K*



Simplicial complex K: a union of simplexes

Overlap of any 2 simplexes produces a simplex from K

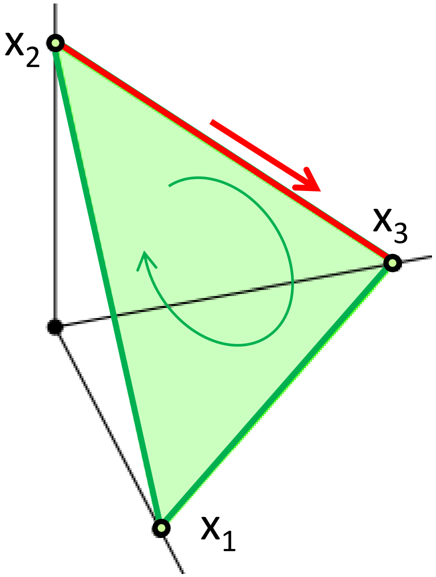




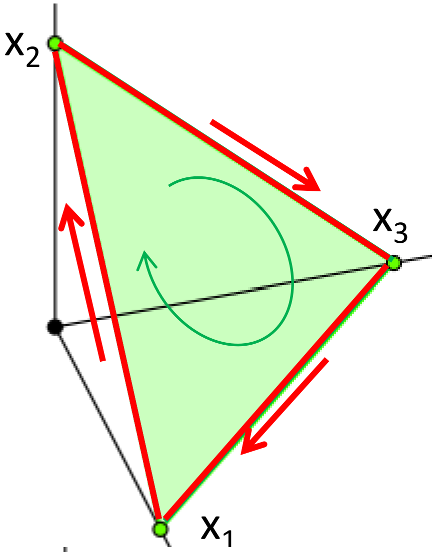
Simplicial complex K is defined by:

1. the list of its simplexes
2. the list of simplex “incidences”

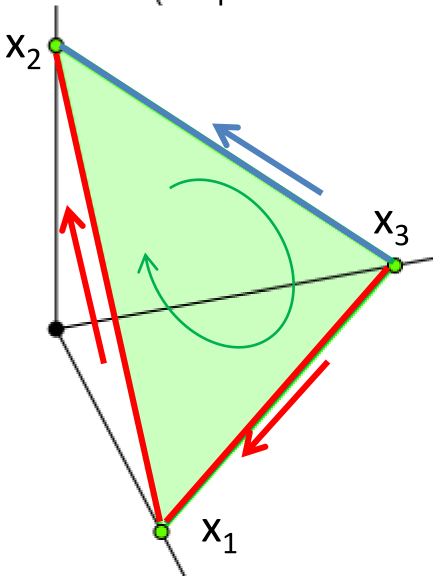
Simplex orientation



Boundary of the simplex

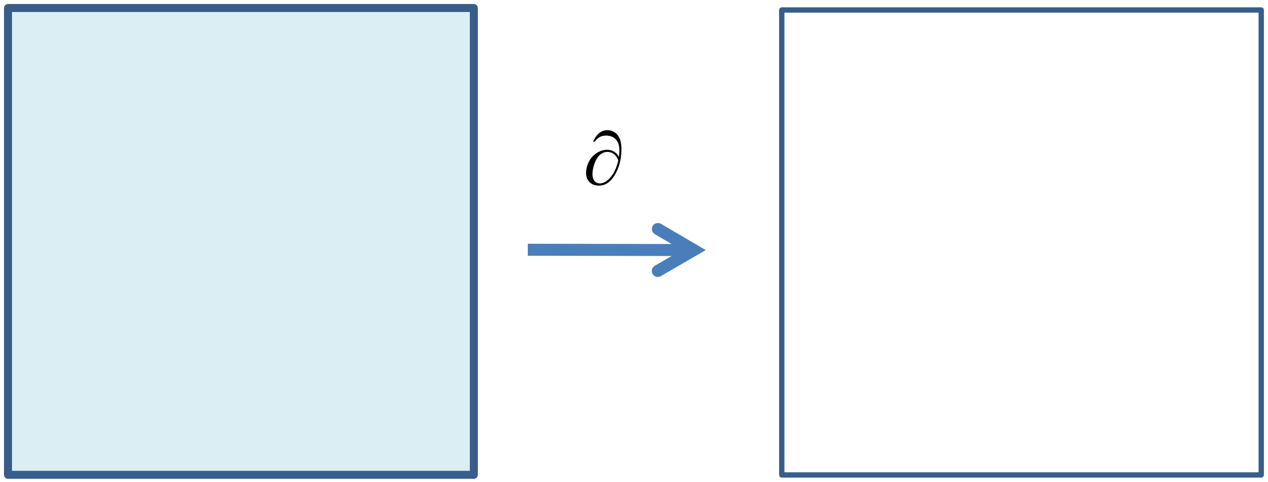


$$\partial \sigma_{123}^{(2)} = \sigma_{12}^{(1)} + \sigma_{23}^{(1)} + \sigma_{31}^{(1)}$$

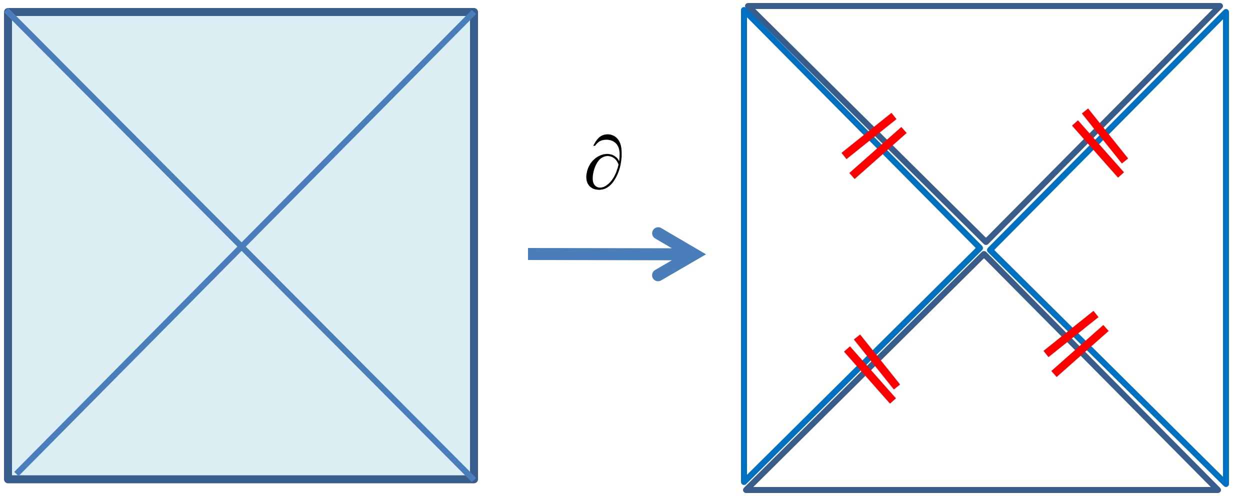


$$\partial \sigma_{123}^{(2)} \neq \sigma_{12}^{(1)} - \sigma_{23}^{(1)} + \sigma_{31}^{(1)}$$

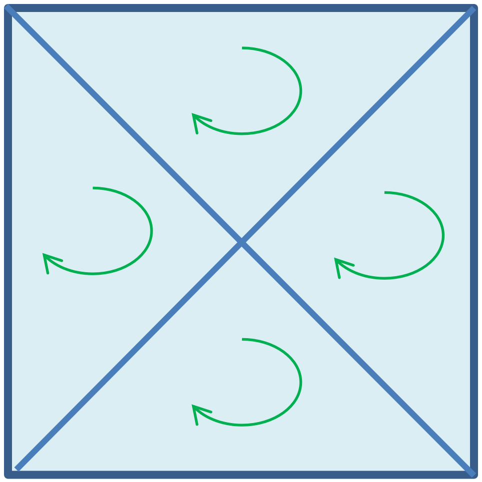
Boundary of a polyhedron



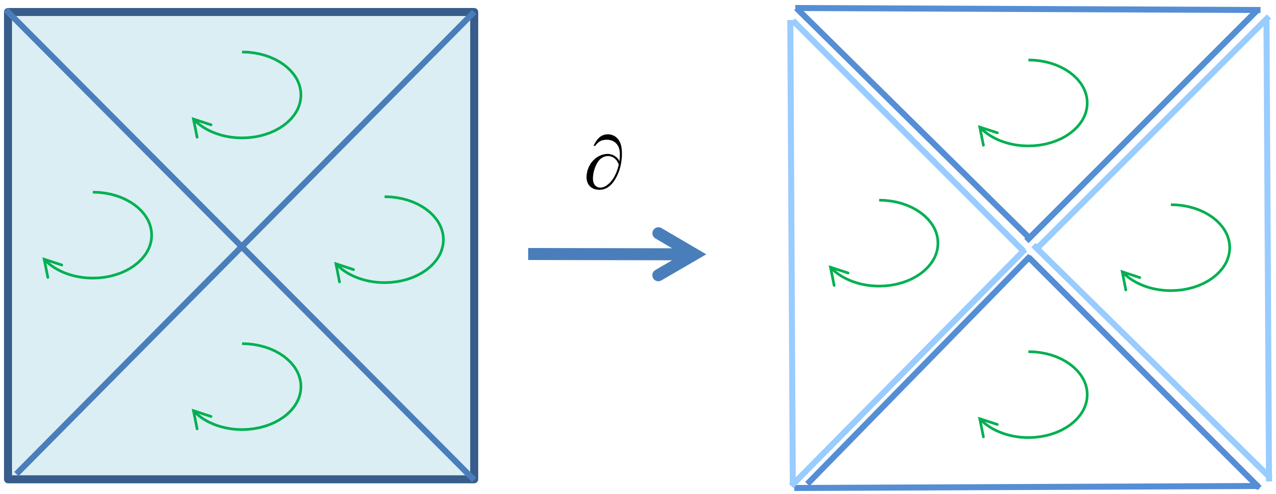
Boundary of the simplicial complex



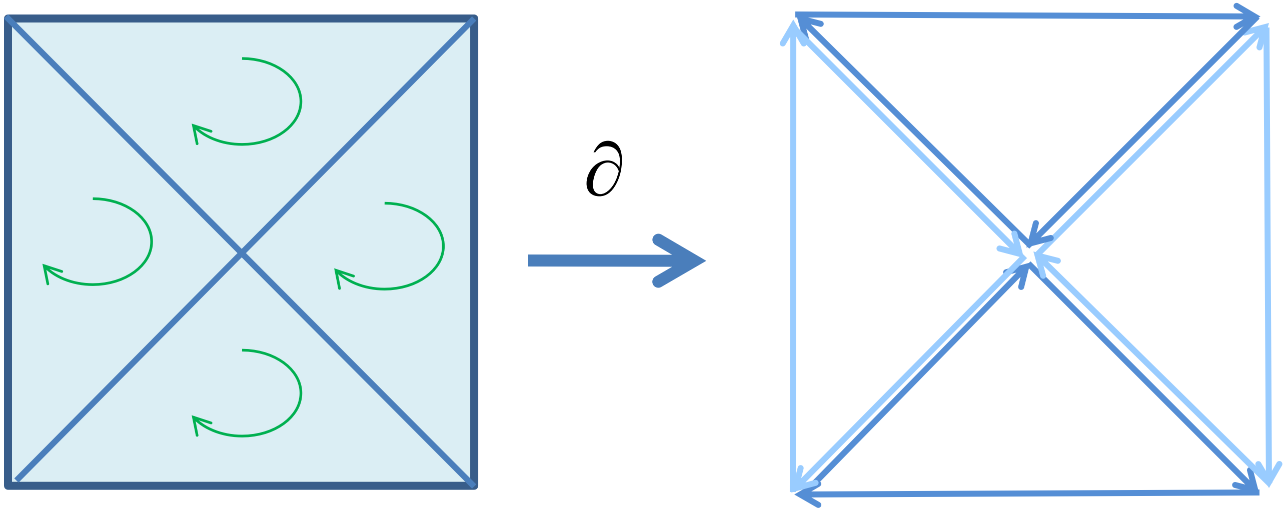
Boundary of an *oriented* simplicial complex



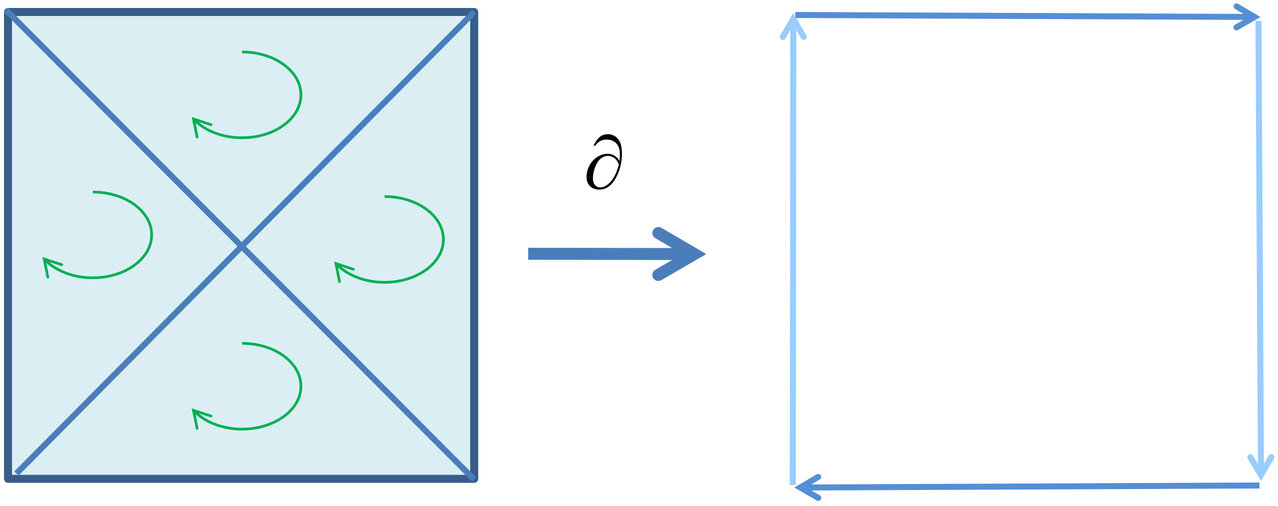
Boundary of an *oriented* simplicial complex



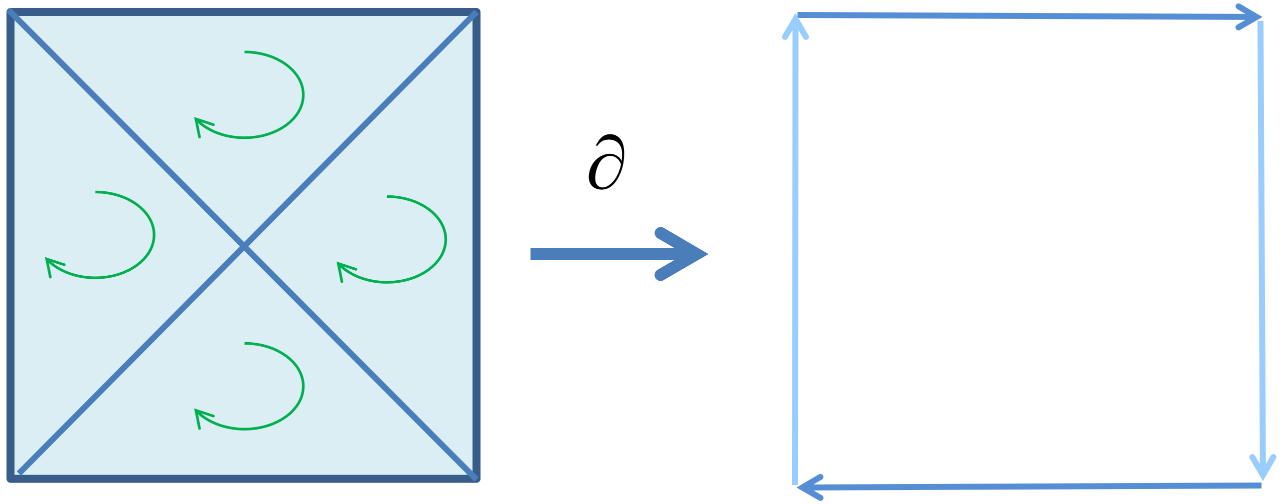
Boundary of an *oriented* simplicial complex

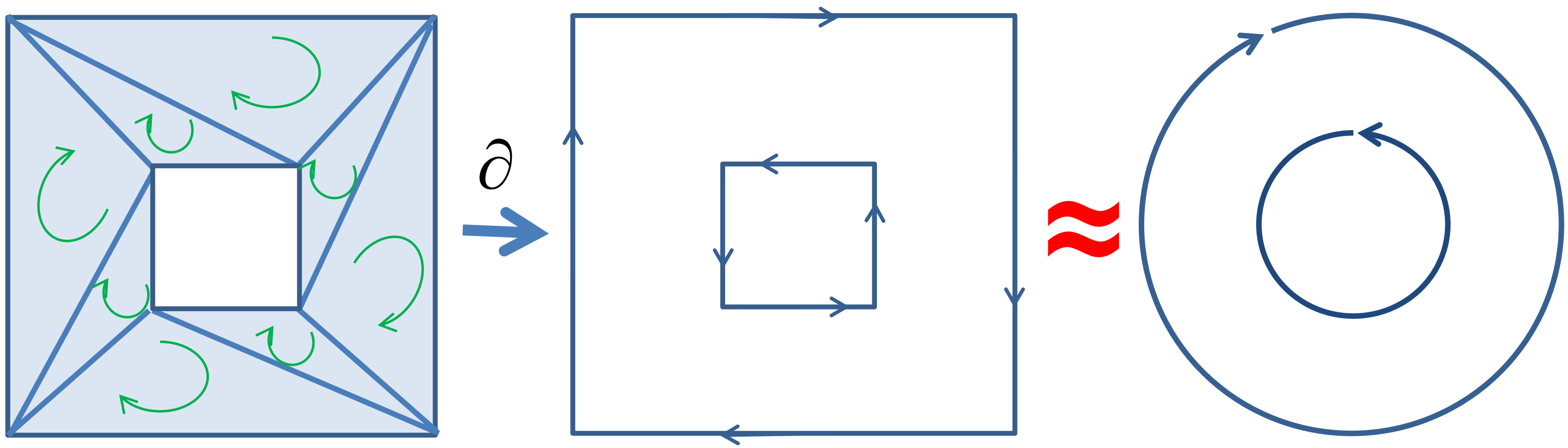


Boundary of an *oriented* simplicial complex



Simplicial complex *K*, orientation





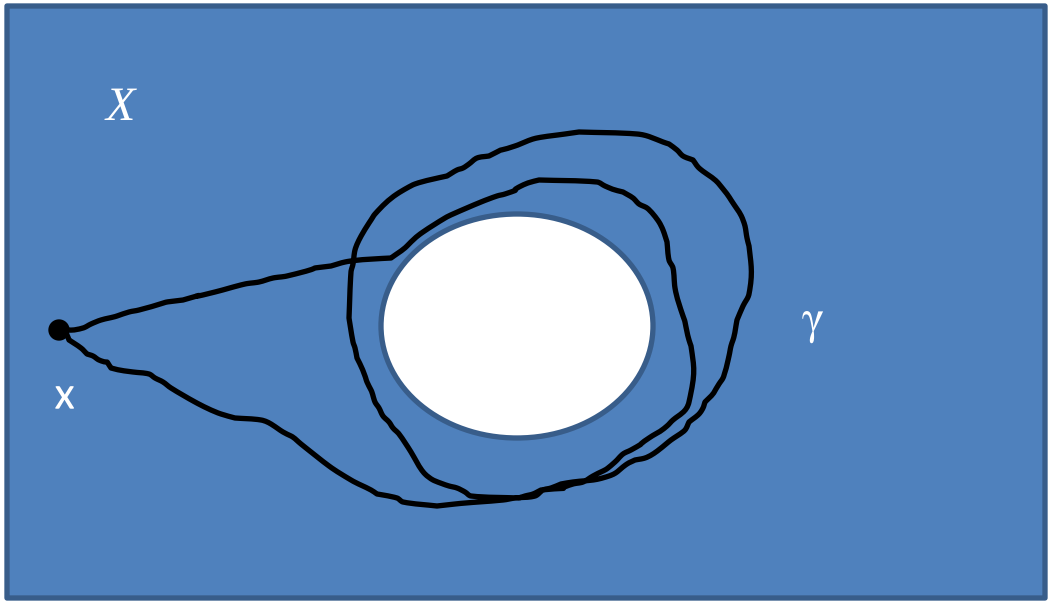
What is the use of simplicial complexes?

$Topology \;\rightarrow$

$Algebra \;\rightarrow$

$Computation$

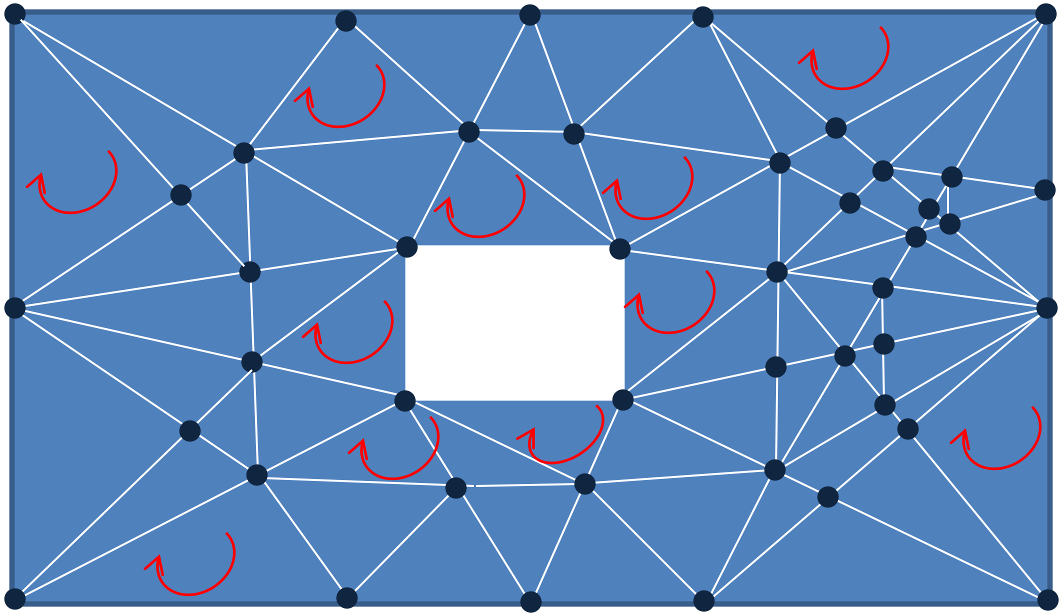
Topology via types of paths



Paths, equivalence classes,[γ]

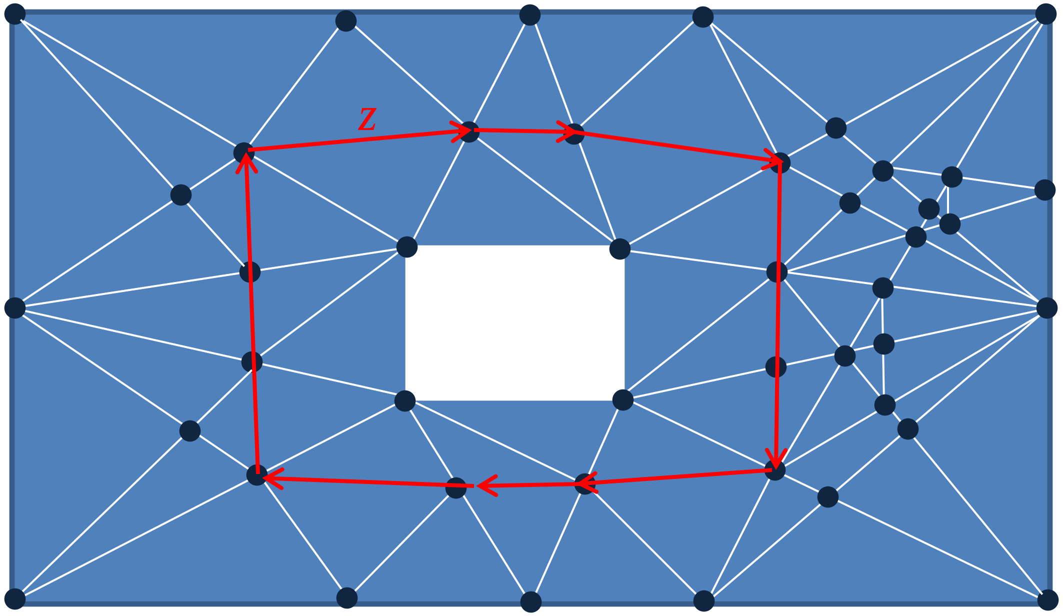
*Topological index m of the path $[\gamma]_m$ produced the **fundamental group $\pi_1(X)= \mathbb{Z}$***

Simplicial complex



Space is approximated by the complex K

Discretization of paths

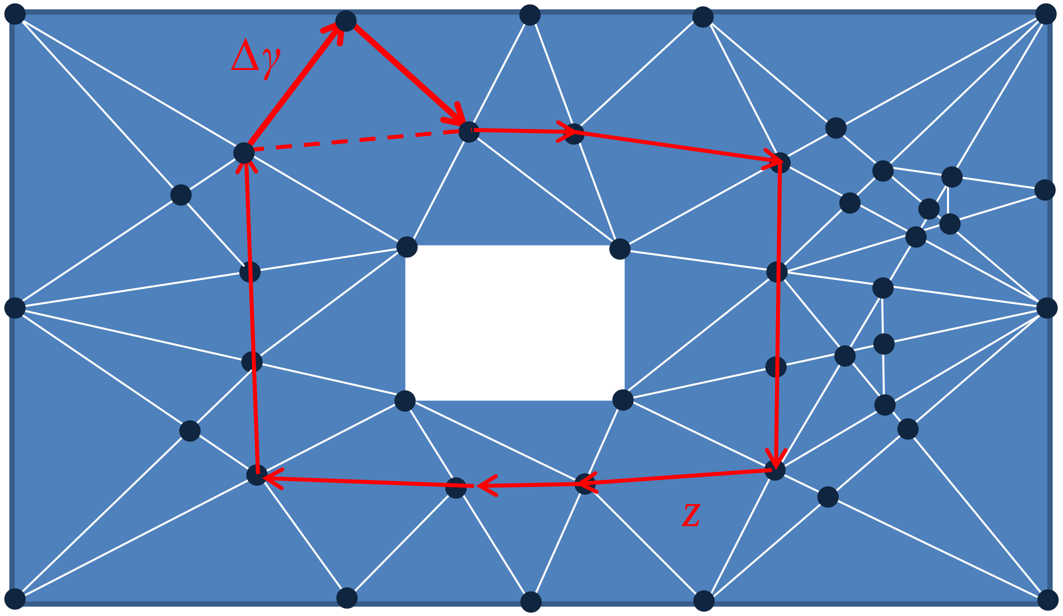


Space is approximated by the complex K

Paths γ are defined ***over K***

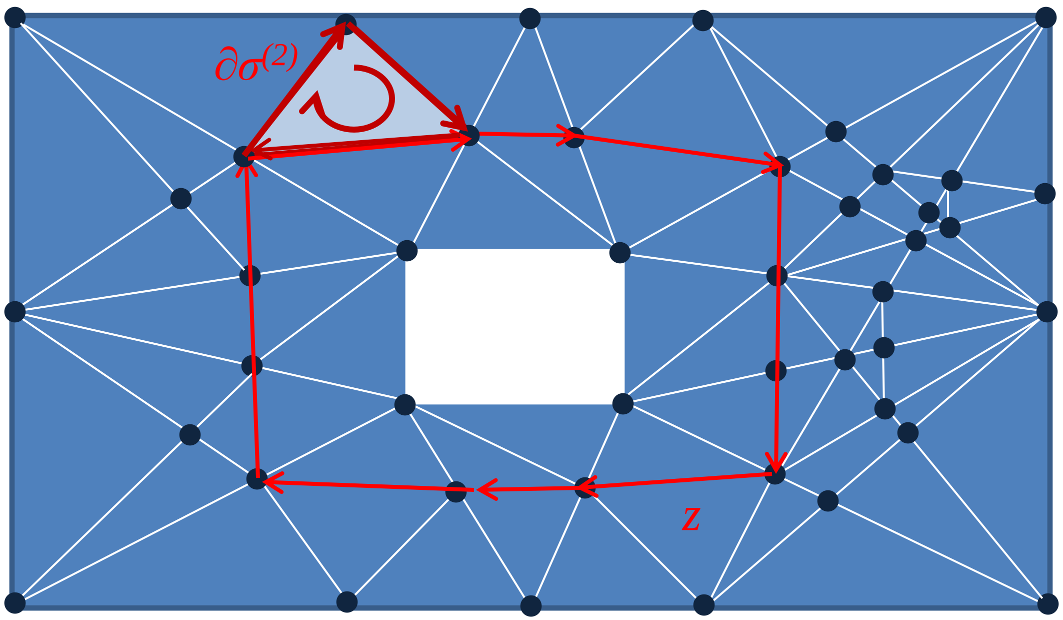
Paths $\gamma \longrightarrow$ cycles, z

Cycle deformation



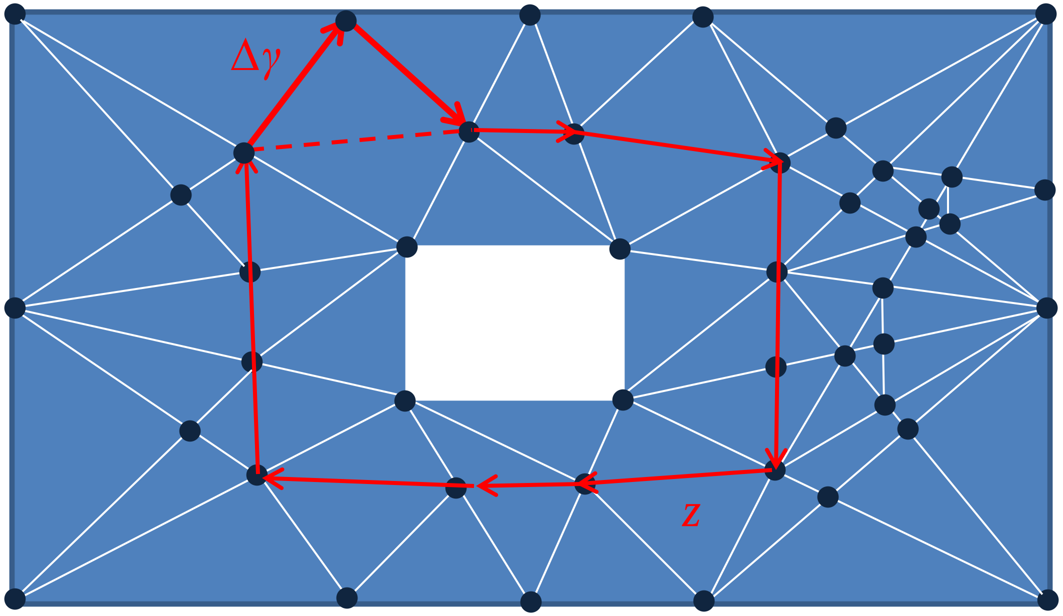
A cycle z can be deformed over the simplex K

Cycle deformation



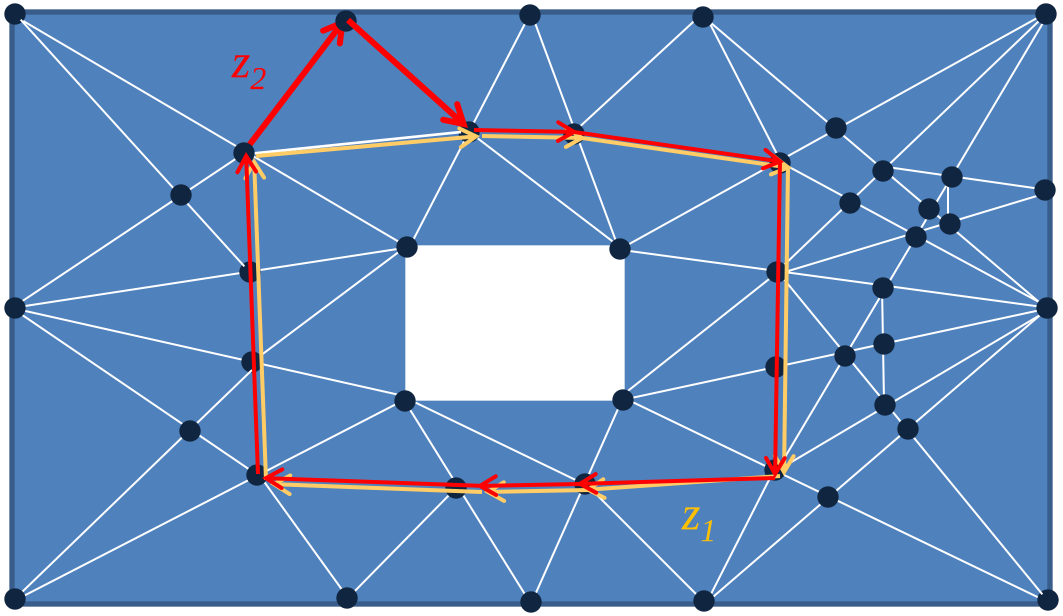
Cycle deformations Δz*'s are snapped over the boundaries of 2D simplexes:* $z_1 = z_2 + \partial\sigma^{(2)}$

Cycle deformation



Cycle deformations Δz*'s are snapped over the boundaries of 2D simplexes:* $z_1 = z_2 + \partial\sigma^{(2)}$

Homologous cycles

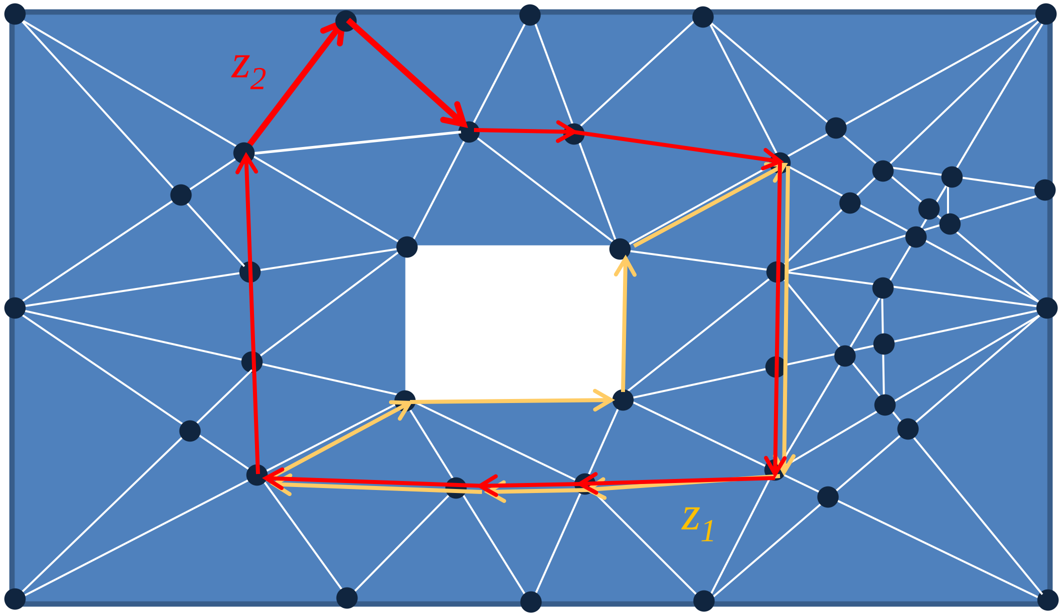


Homologous cycles: $z_1 \sim z_2$ (homotopic paths)

Topological analysis with cycles

How many classes of homologous cycles are there?

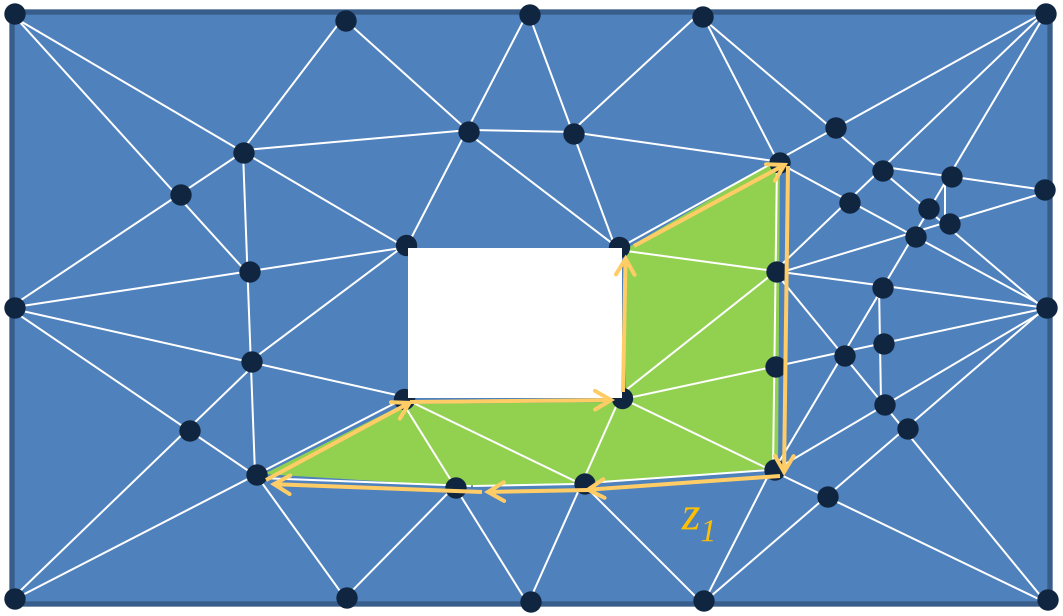
Non-homologous cycles



Non-homologous cycles: $z_1 \nsim z_2$

What feature makes these cycles different?

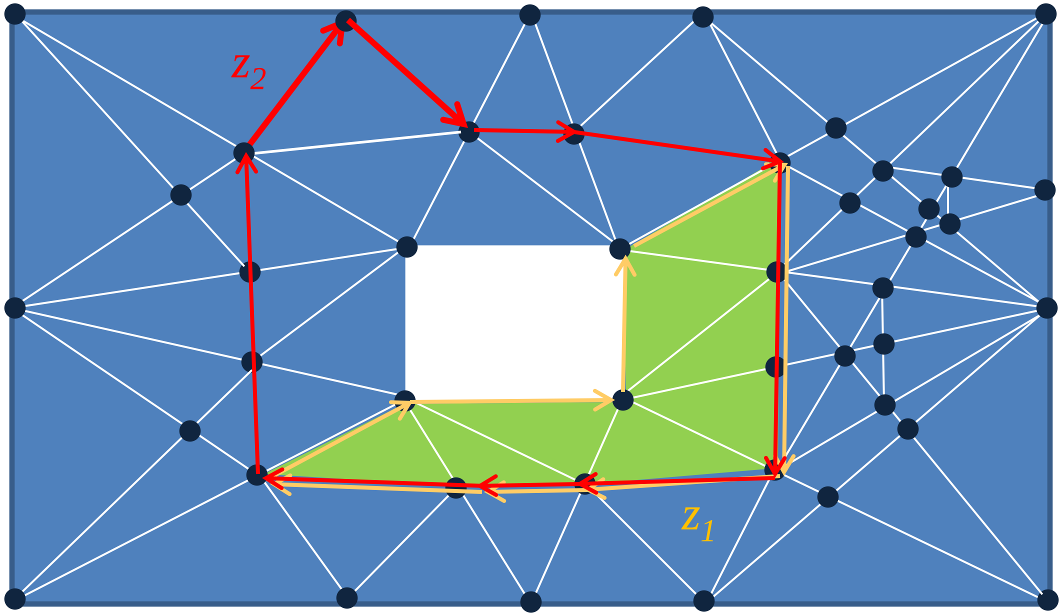
Classes of cycles



Cycles z_2 can be contracted to a point,

because it is a boundary of a ***contractible*** 2D “surface”

Classes of cycles



Non-homologous cycles: $z_1 \nsim z_2$

1. Contractible cycles, e.g. z_1
2. Non-contractible cycles, e.g. z_2

Homologies

$$H_1 = \left(\text{classes of homologous cycles}\right)$$

"z_1 is homologous to z_2" =

= "z_1 is equal to z_2 modulo a boundary cycle"

$$H_1 = (Cycles)\big/\left(Boundaries\right)$$

Homologies

"z_1 is homologous to z_2" =

= "z_1 is equal to z_2 modulo a boundary cycle"

$$H_1 = (Cycles)/(Contractible\ cycles)$$

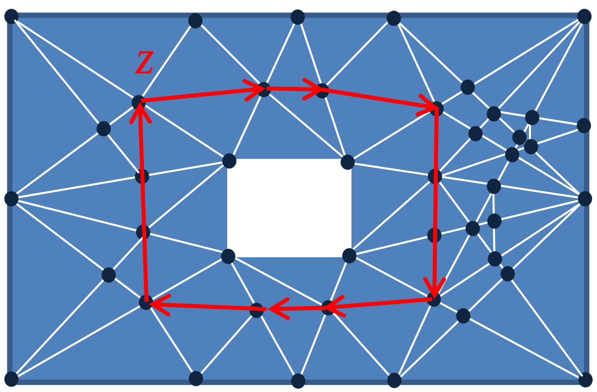
$$= (\text{non-contractible cycles and their multiples})$$

Homologies

"z_1 is homologous to z_2" =

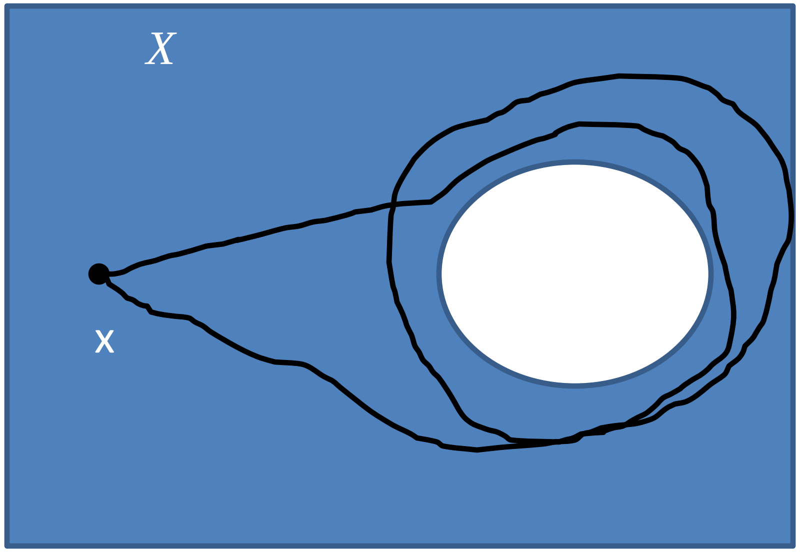
= "z_1 is equal to z_2 modulo a boundary cycle"

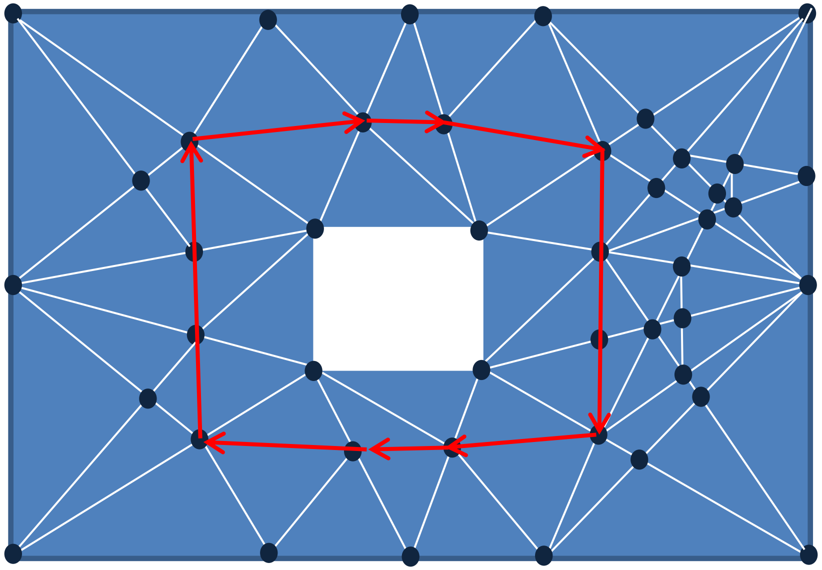
$$H_1 = (Cycles) \big/ \big(Contractible\ cycles\big) = \mathbb{Z}$$



First homology group

Fundamental vs. homological group





$(paths) \rightarrow (indexes)$

$(cycles) \rightarrow (indexes)$

Fundamental group,

$$\pi_1(X) = \mathbb{Z}$$

First homology group,

$$H_1(X) = \mathbb{Z}$$

Homologies

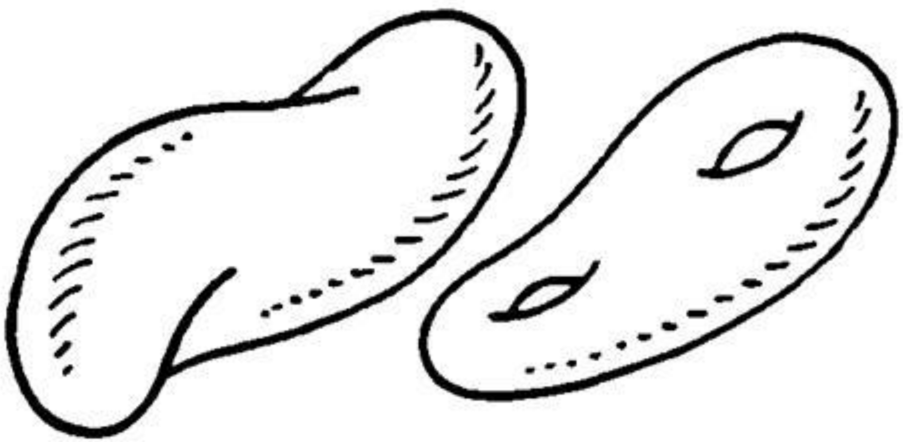
Theorem 1: Homological groups do not depend on simplicial subdivision of polyhedron*

Theorem 2: Homological groups are topologically invariant

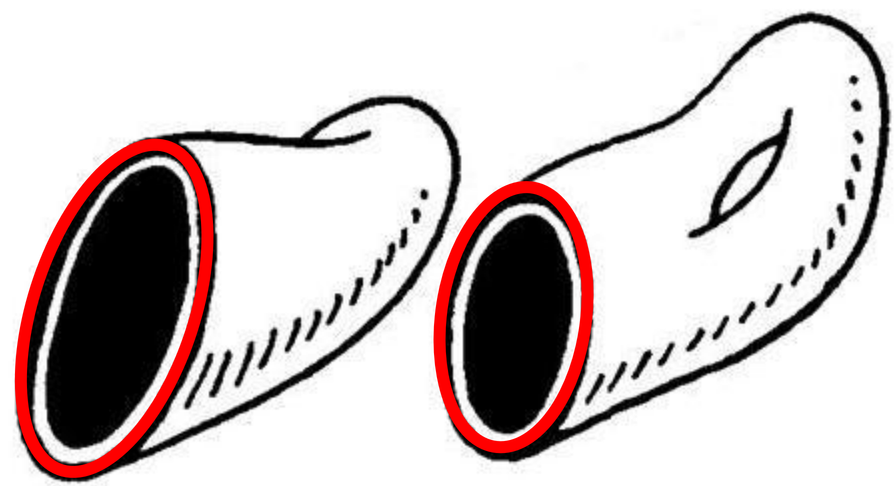
* For fine enough subdivisions

What we ultimately want with homologies

How many classes of homologous cycles are there, in every dimension?



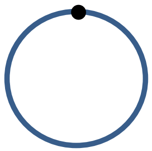
Cycles



Boundaries

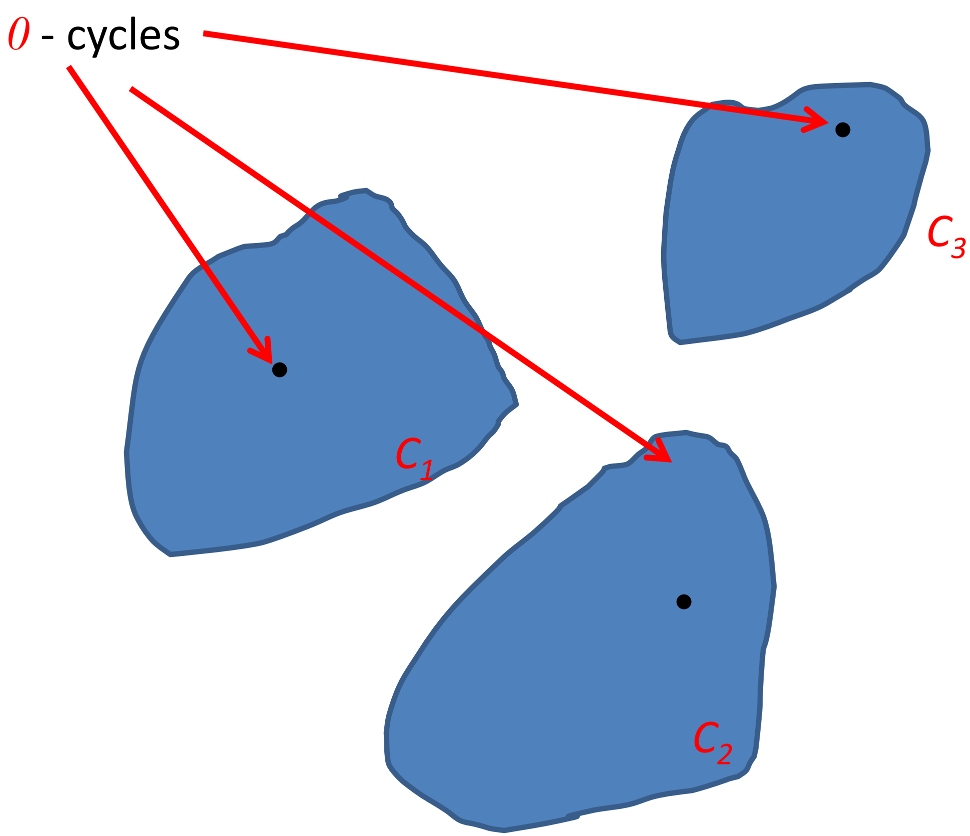
Betti numbers – number of cycles in every dimension

Circle



(1, 1, 0, 0,…)

Topological properties, examples

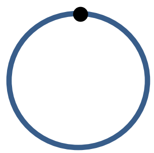


Cycle connectedness: 3 0-cycles, and 3 pieces

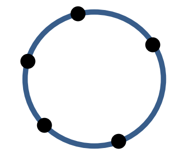
Betti index – base cycles in every dimension

"Topological barcode"

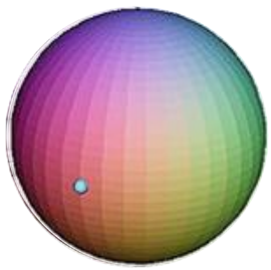
Circle



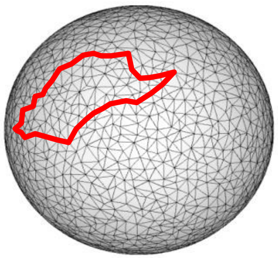
(1, 1, 0, 0,…)



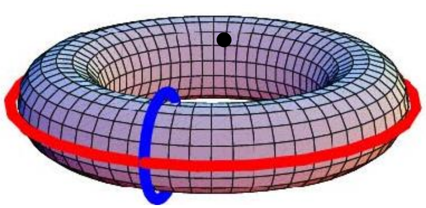
Sphere



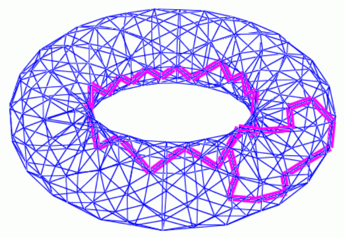
(1, 0 , 1, 0,…)



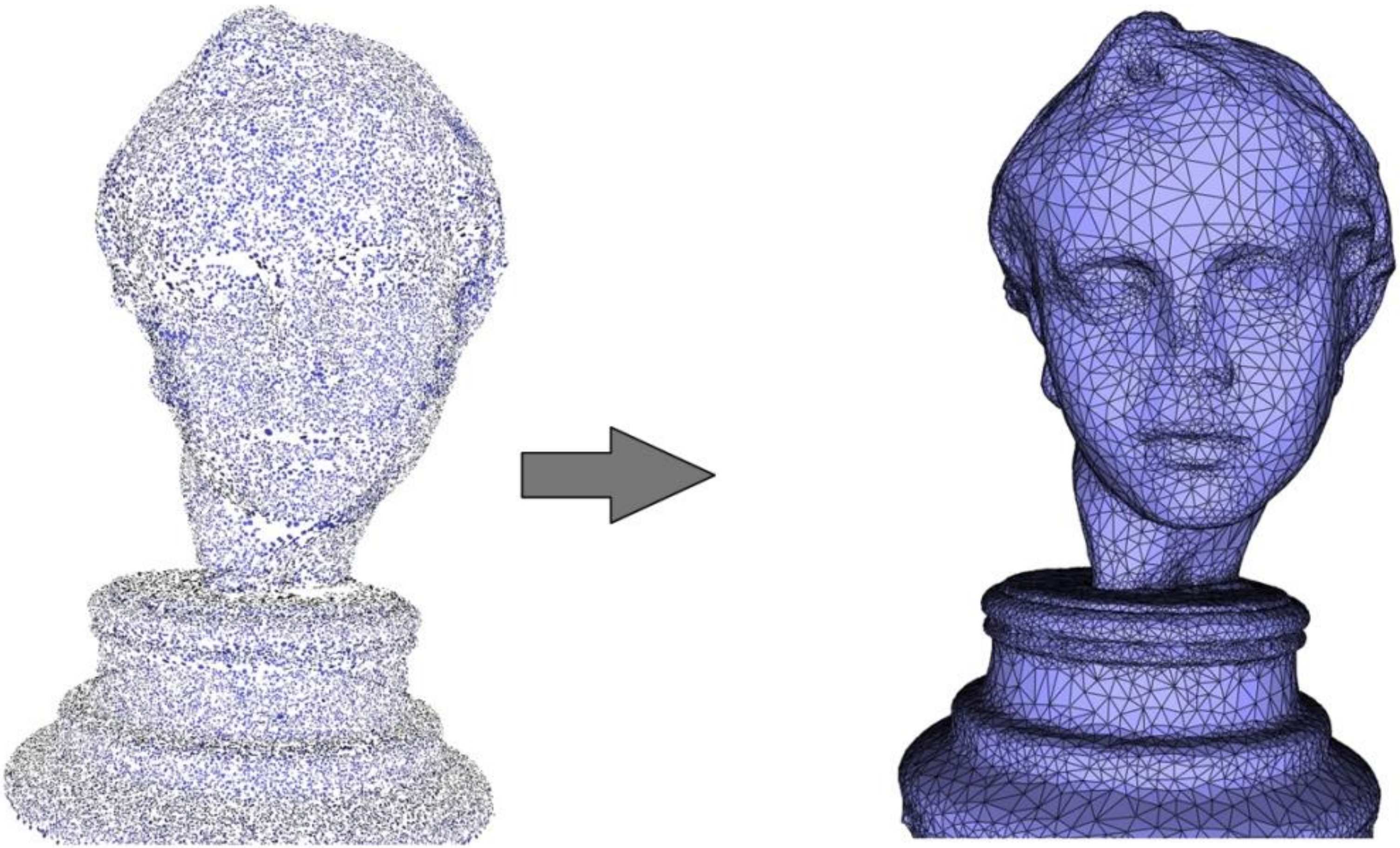
Torus



(1, 2 , 1, 0,…)



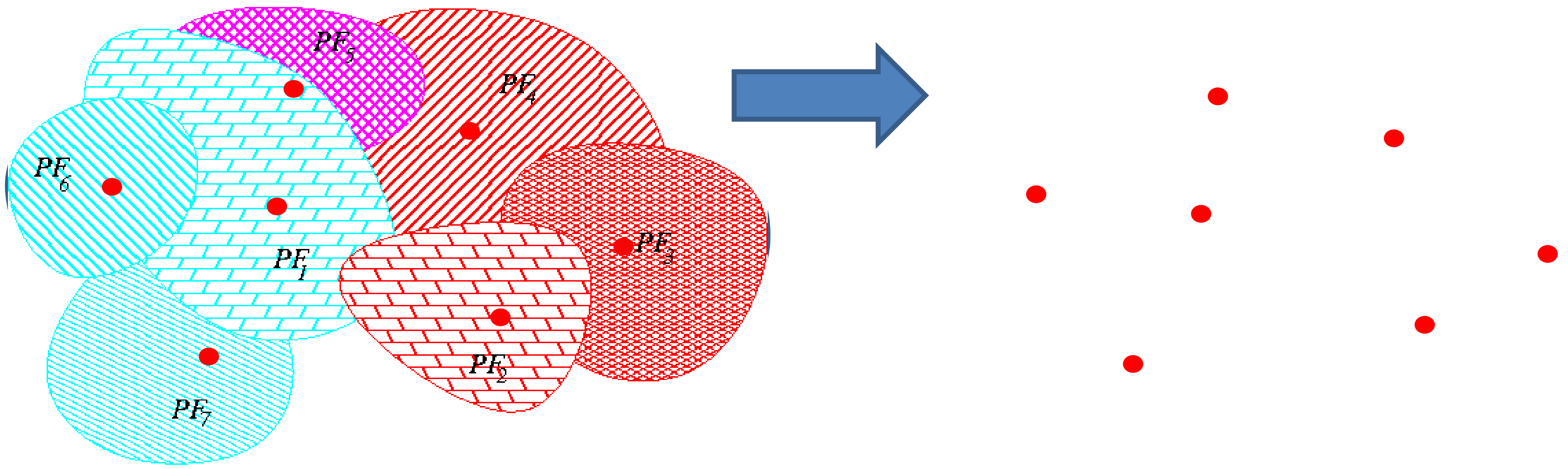
How to build simplexes in practice?



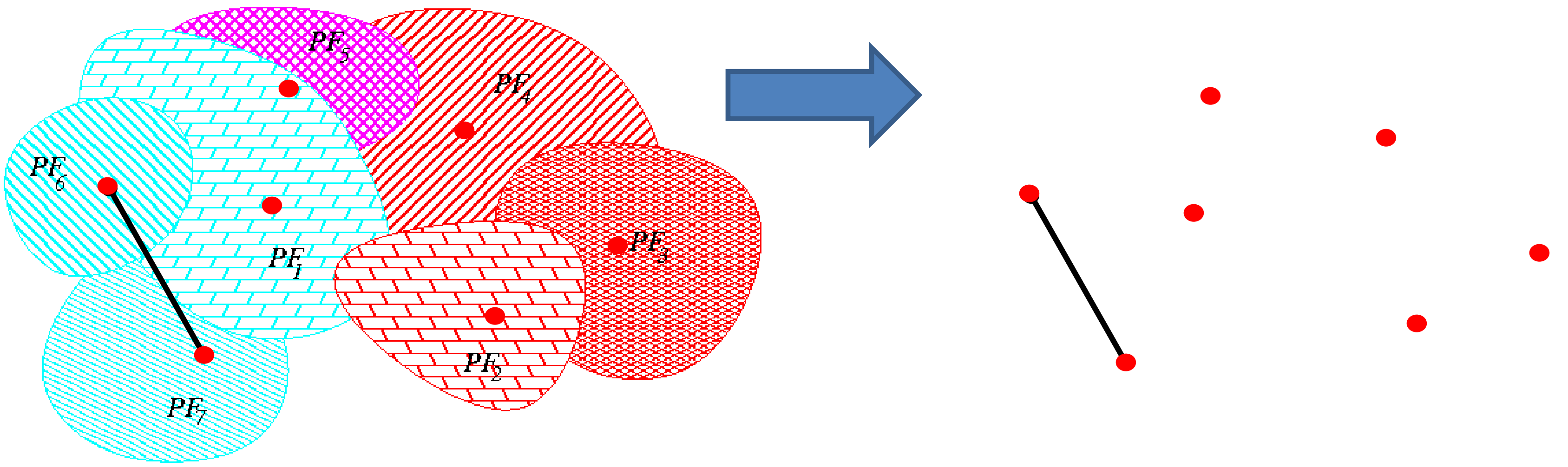
http://www.cgal.org

How to build a triangulation of a surface?

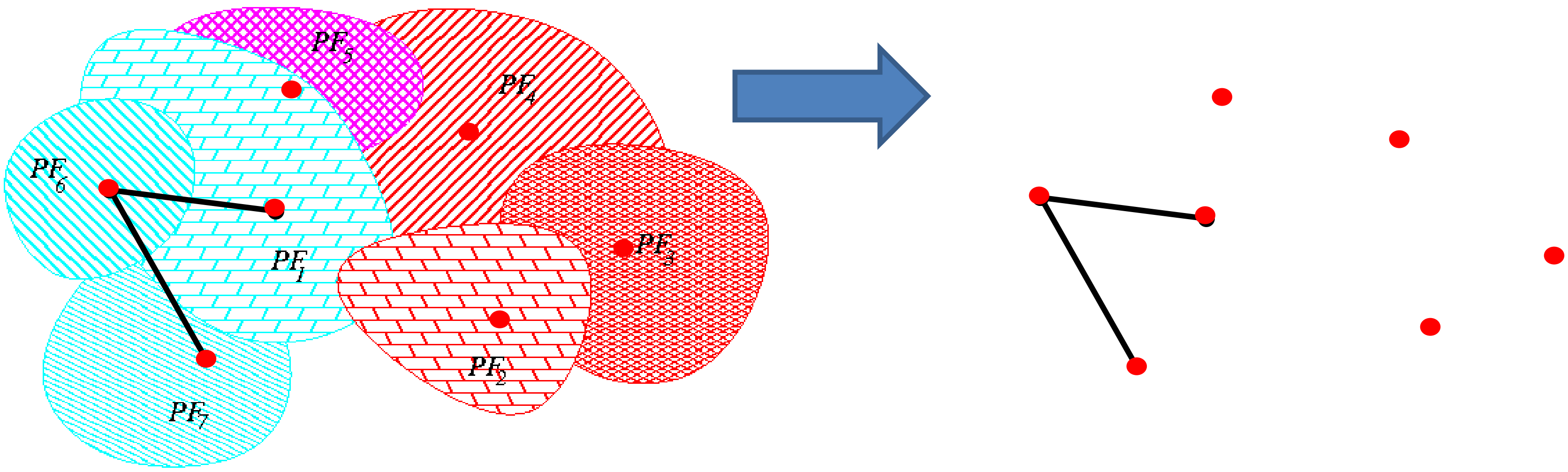
Čech complex



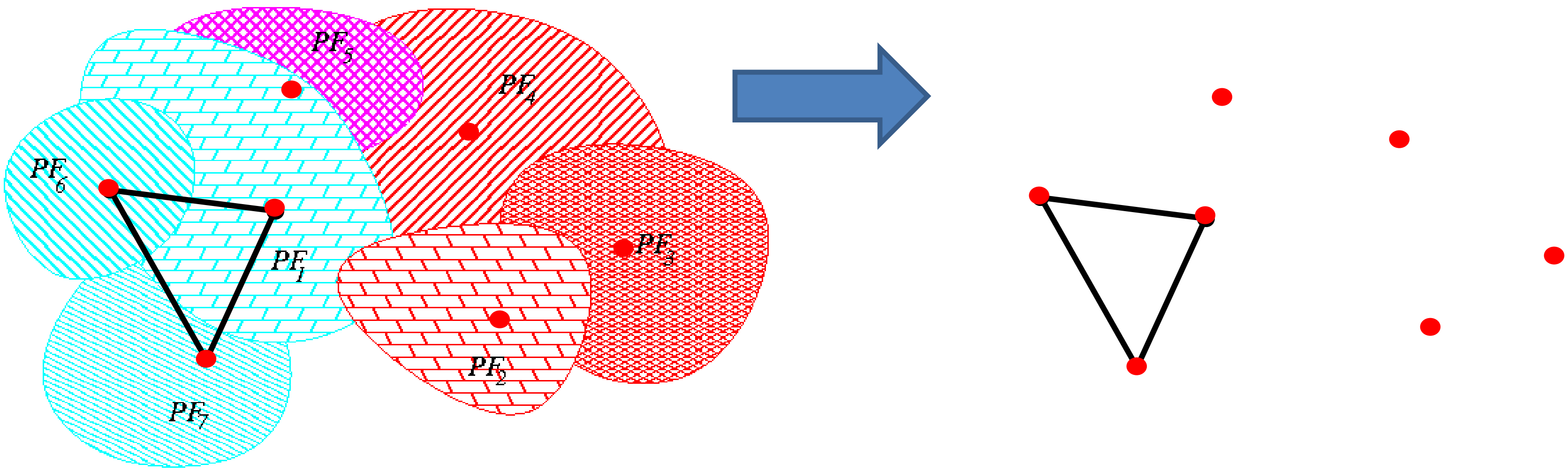
Čech complex



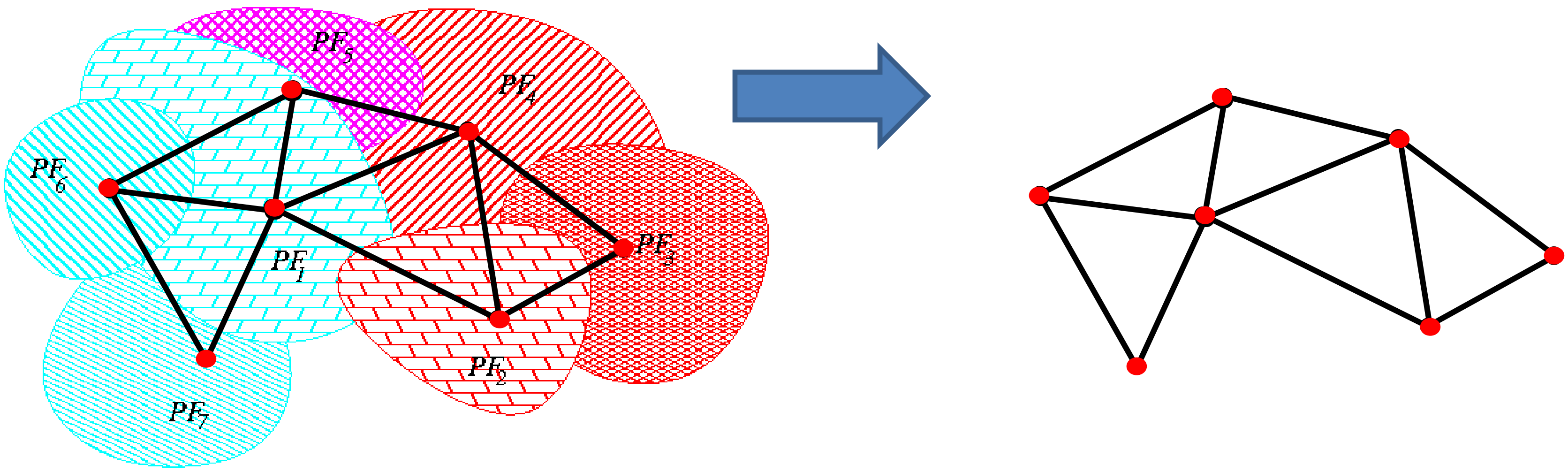
Čech complex



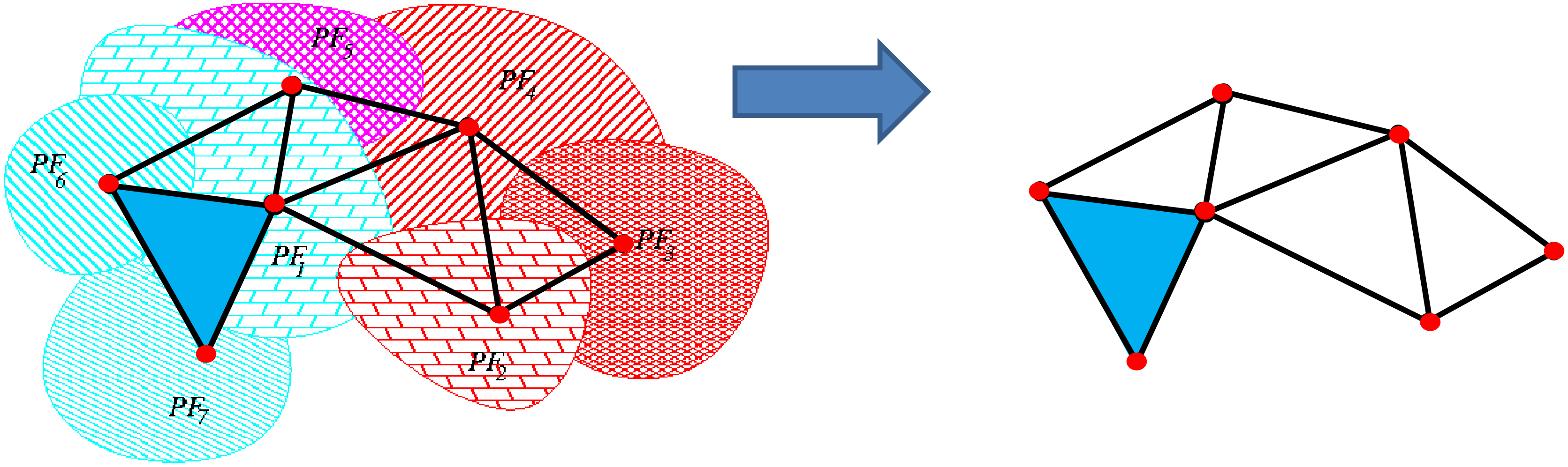
Čech complex



Čech complex

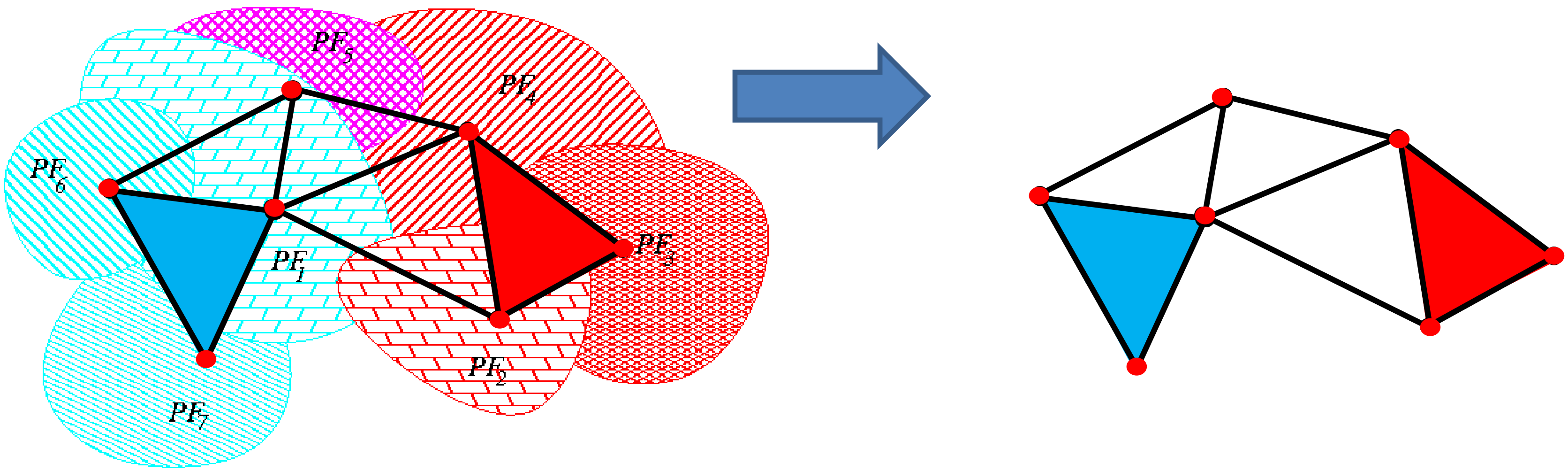


Čech complex



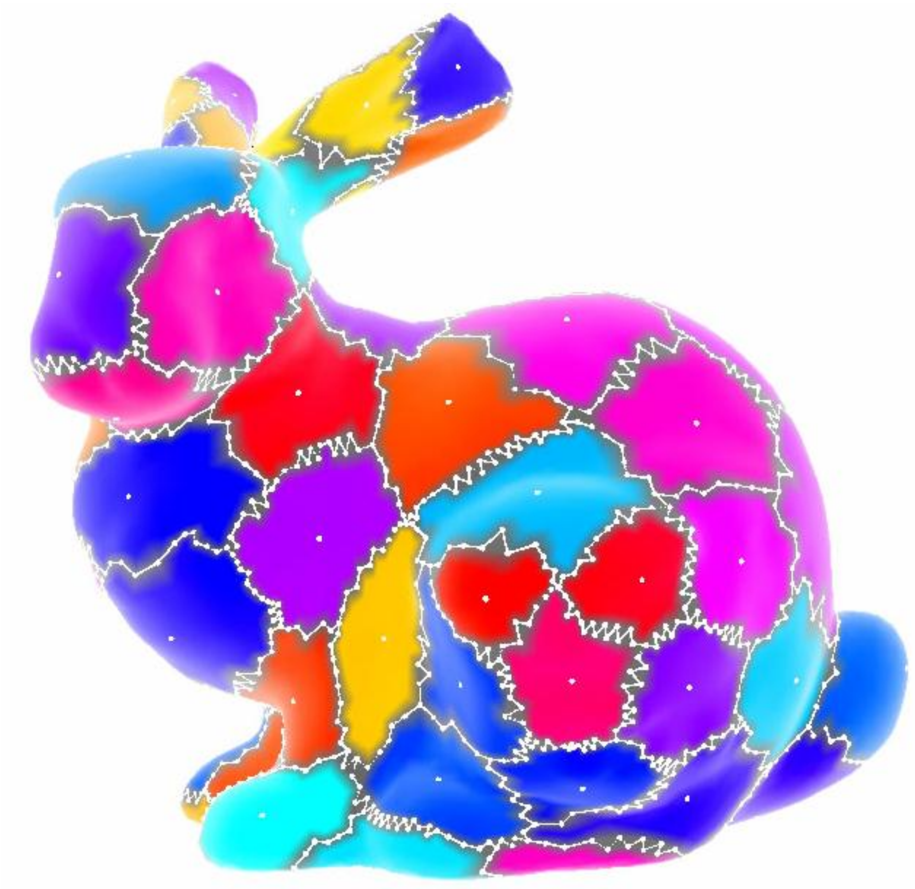
Čech complex

Simplicial complex, K

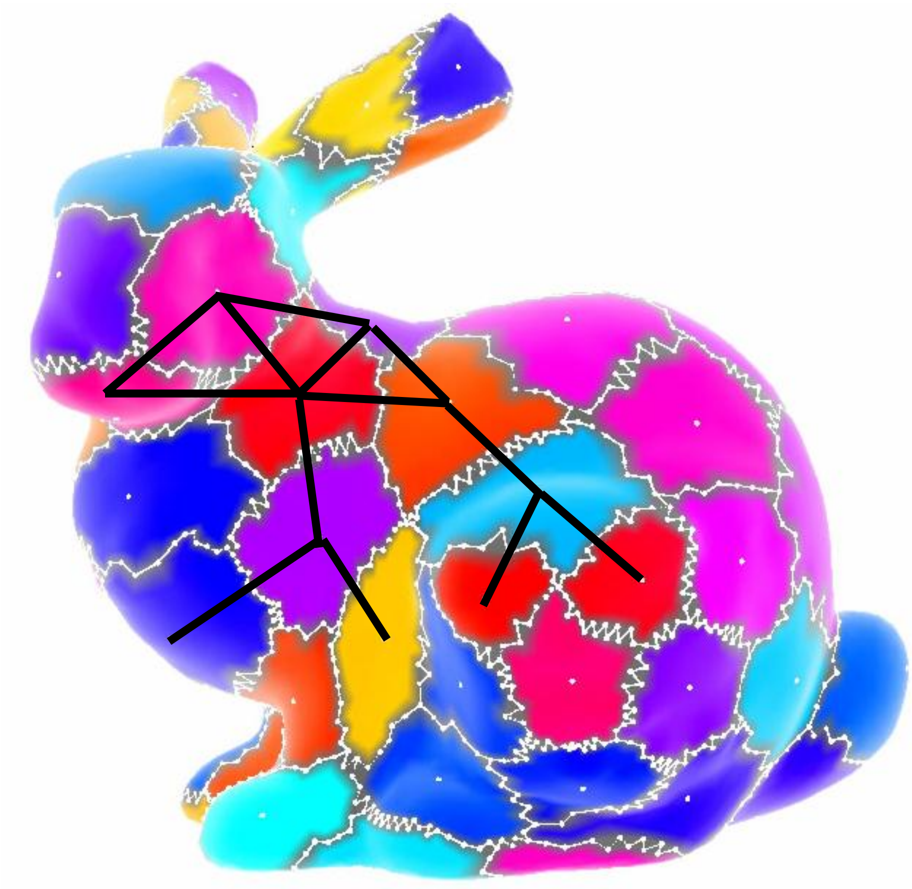


If the cover is fine enough, the homologies of the complex K are the same as the homologies of the original space.

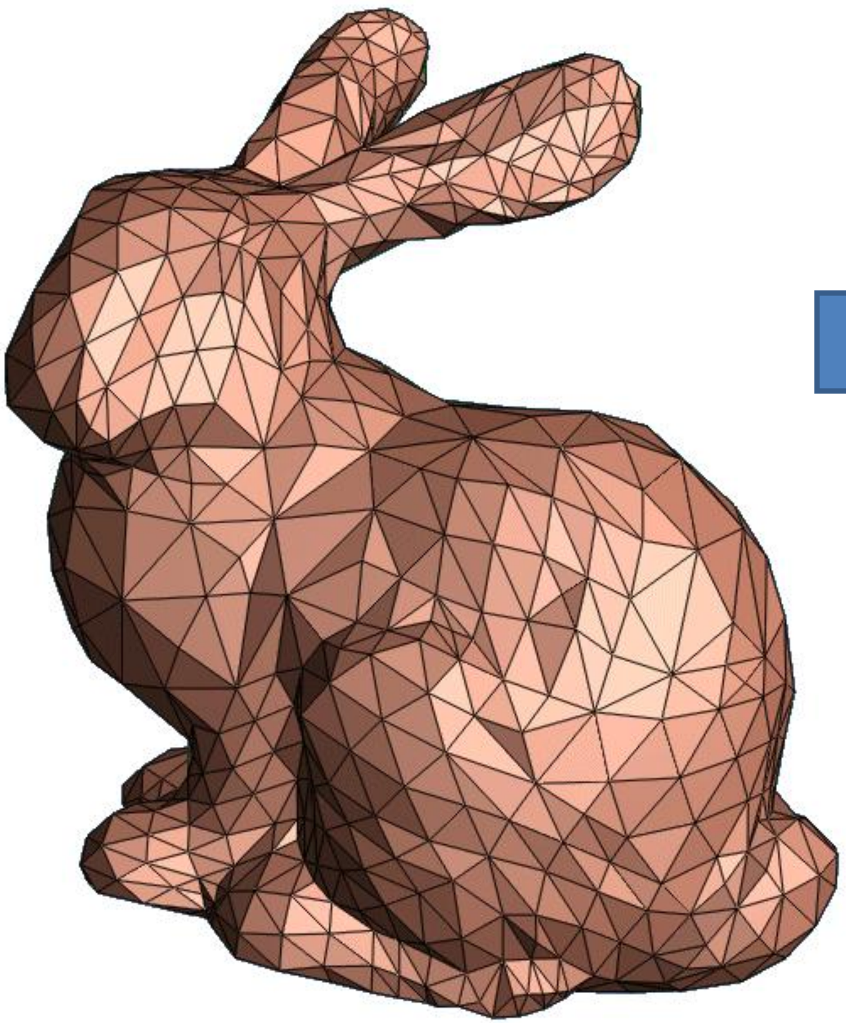
A manifold and its cover



A cover generates simplex

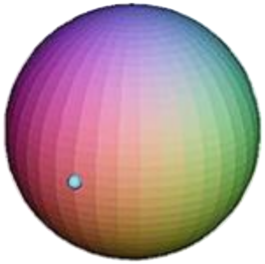


Simplex produces full topological information

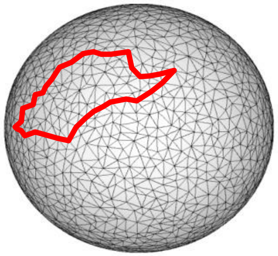


Homologies, etc.

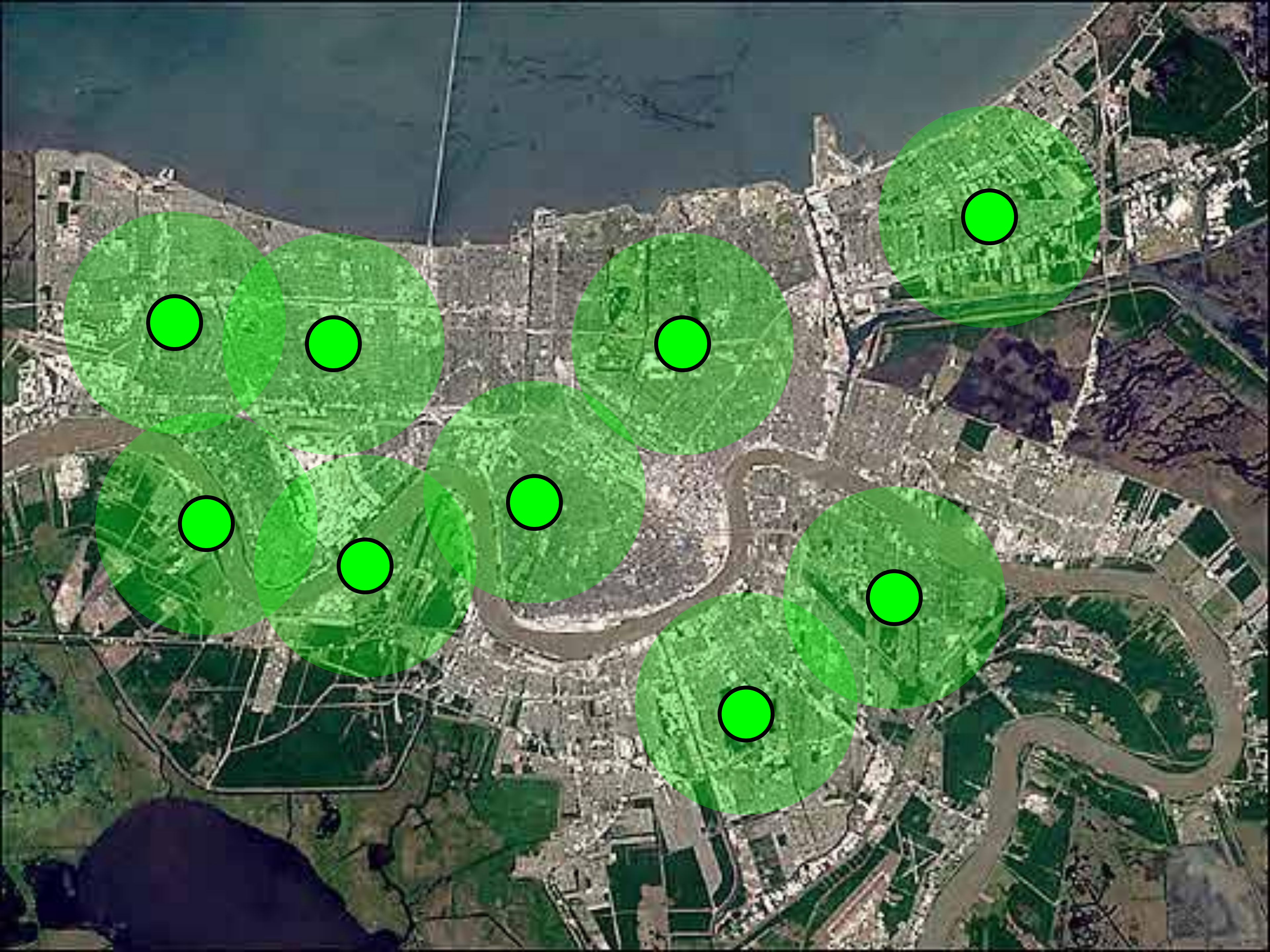
Test: what is the
“Topological barcode” of this space?



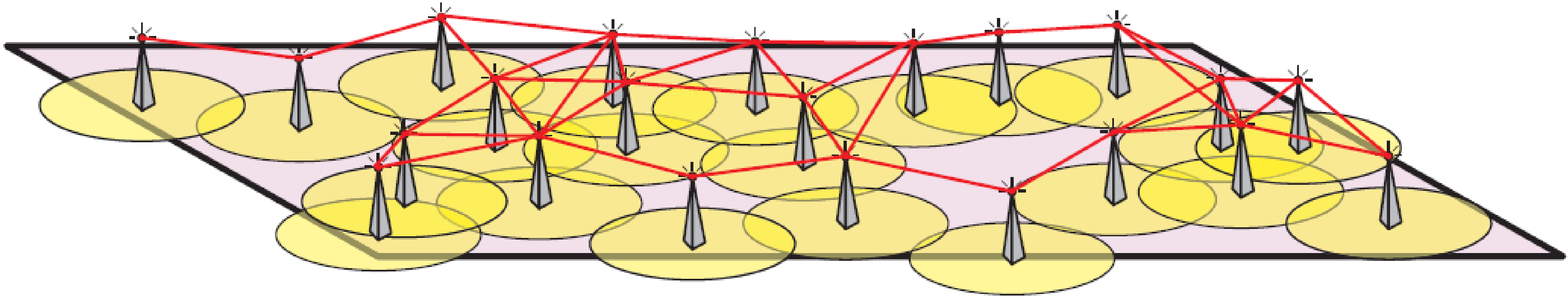
Sphere



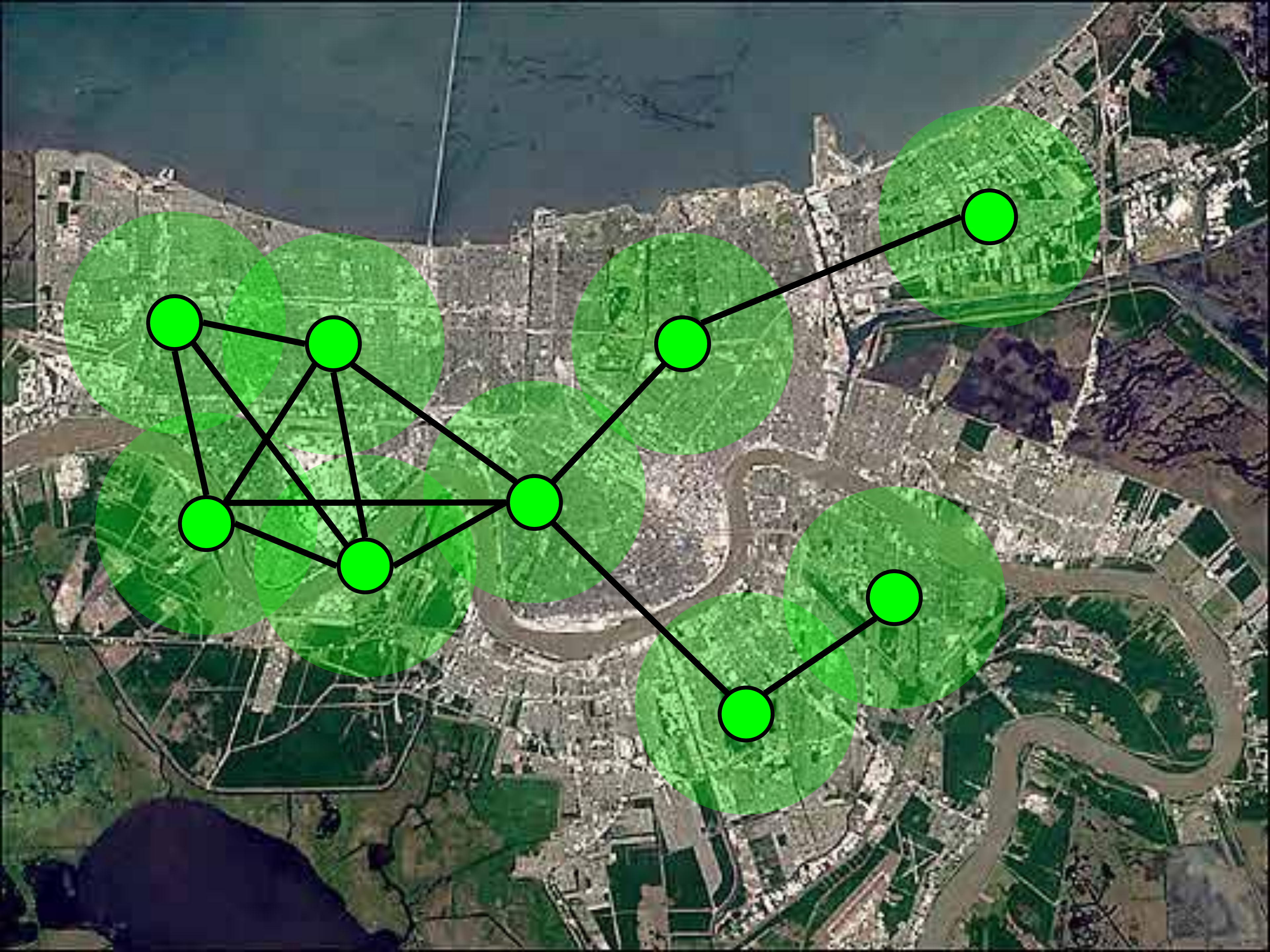
(1, 0 , 1, 0,…)

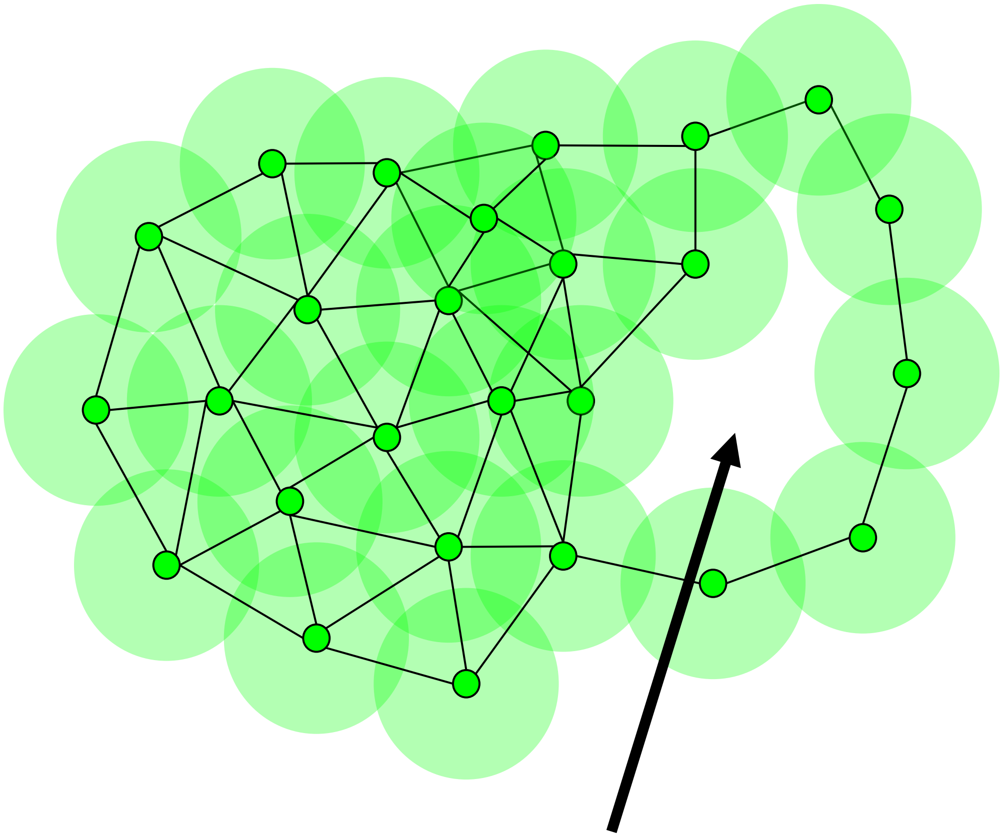


Topology from sensor networks



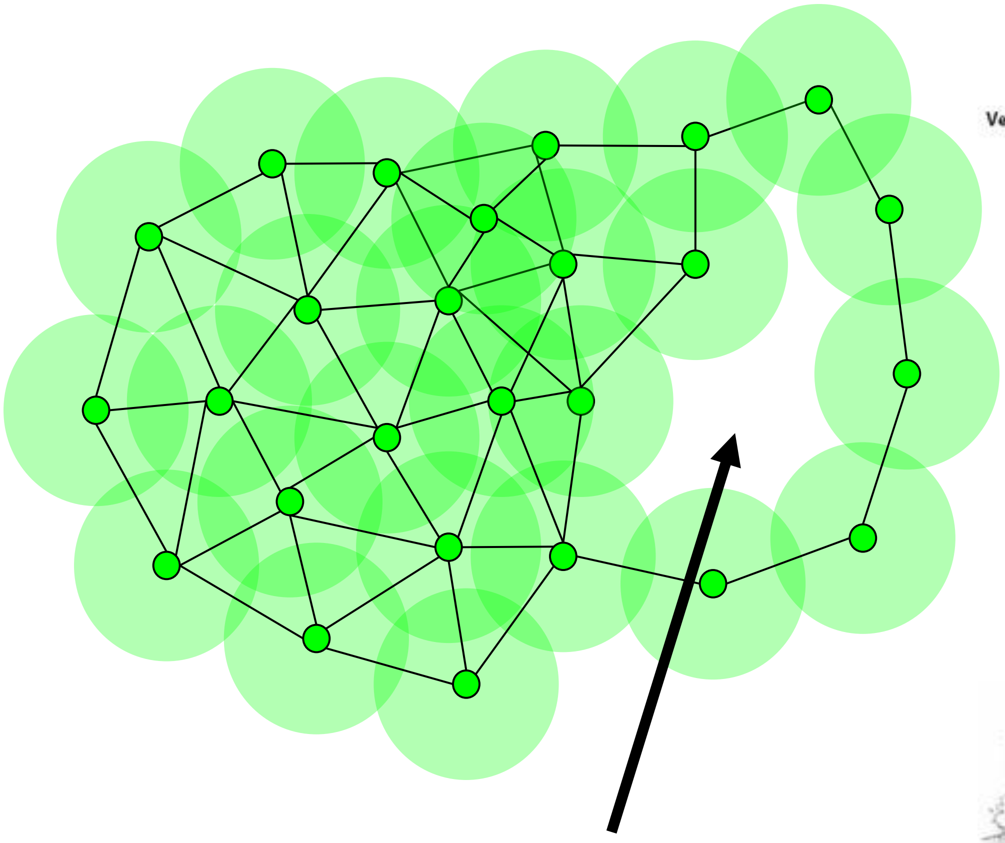
V. de Silva, Homological sensor networks, (2007)





Hole in sensor coverage area

What is the wireless topology of the US?



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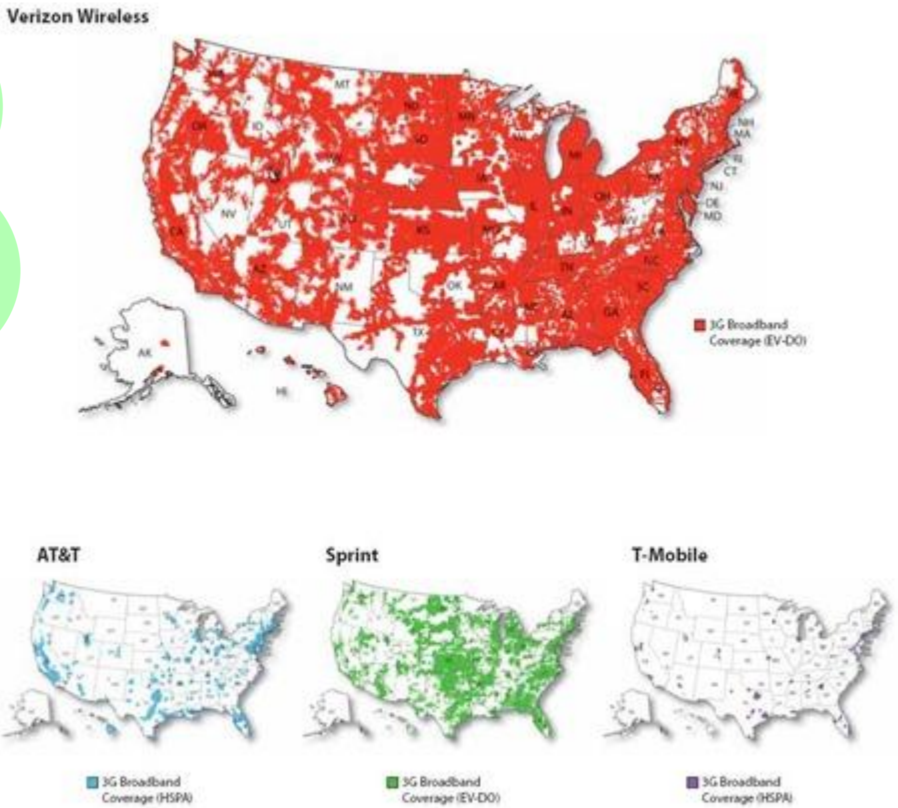
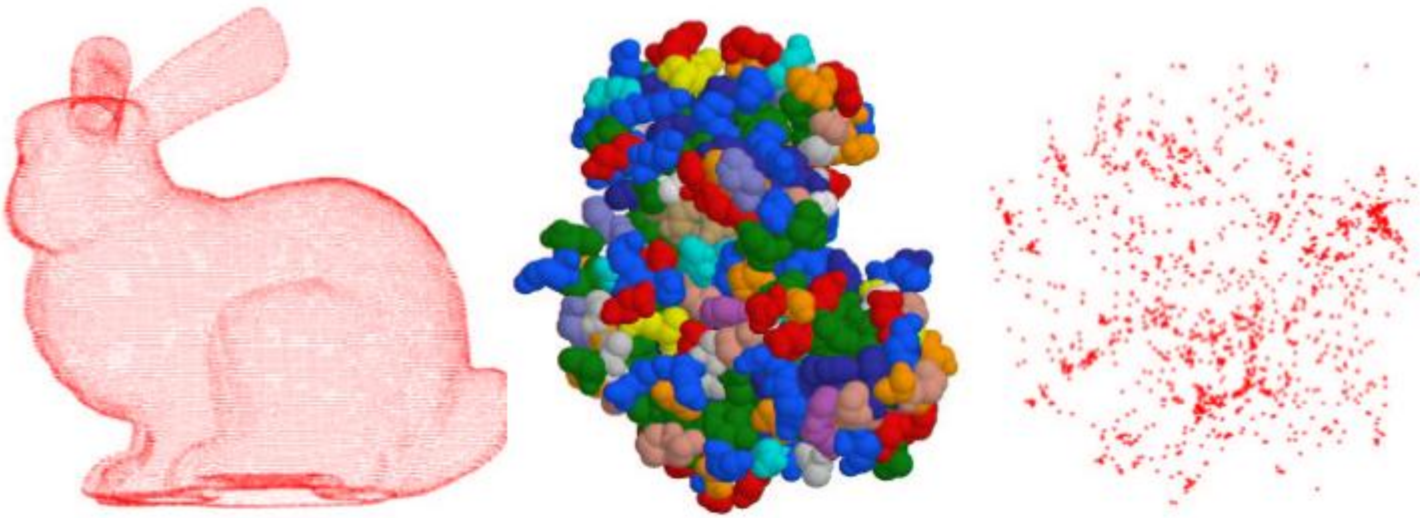
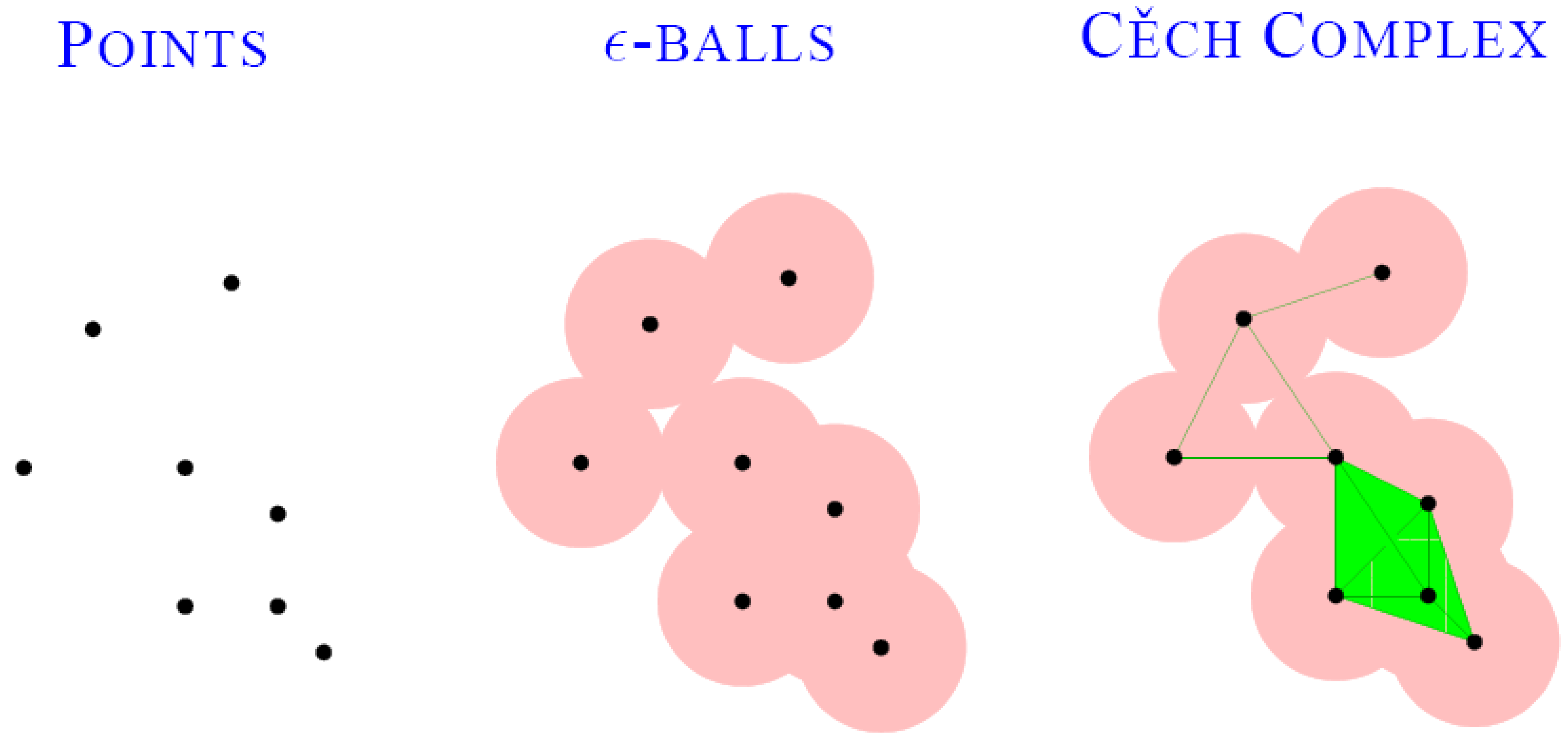


Point cloud data



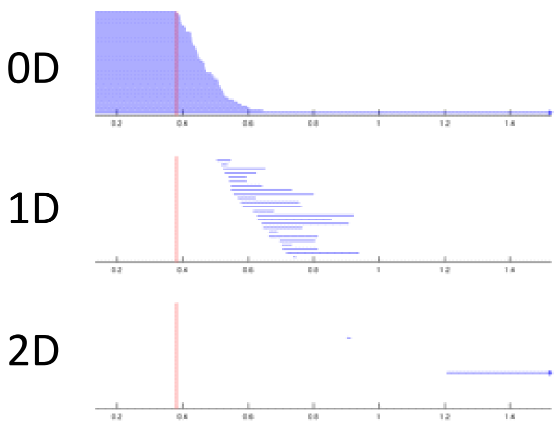
(a) Surface (b) Molecule (c) Universe

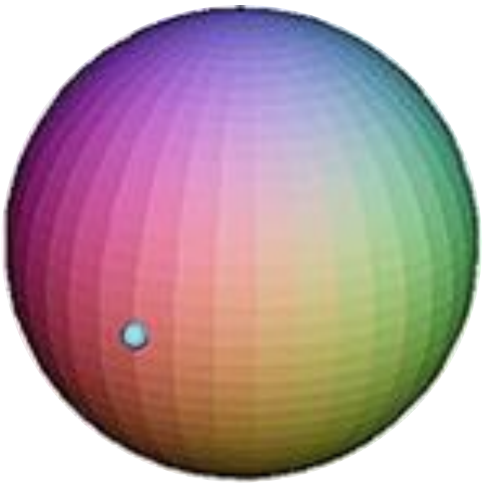
The ideas of topological persistence

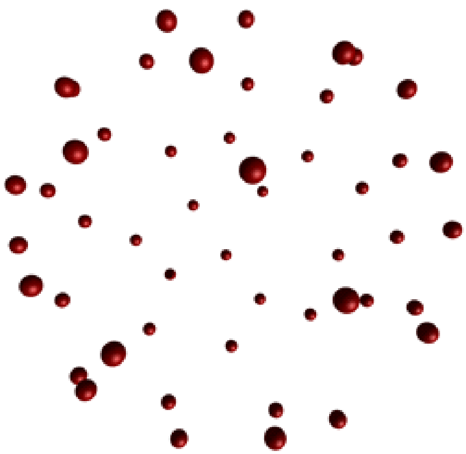


The unfolding of the topological information

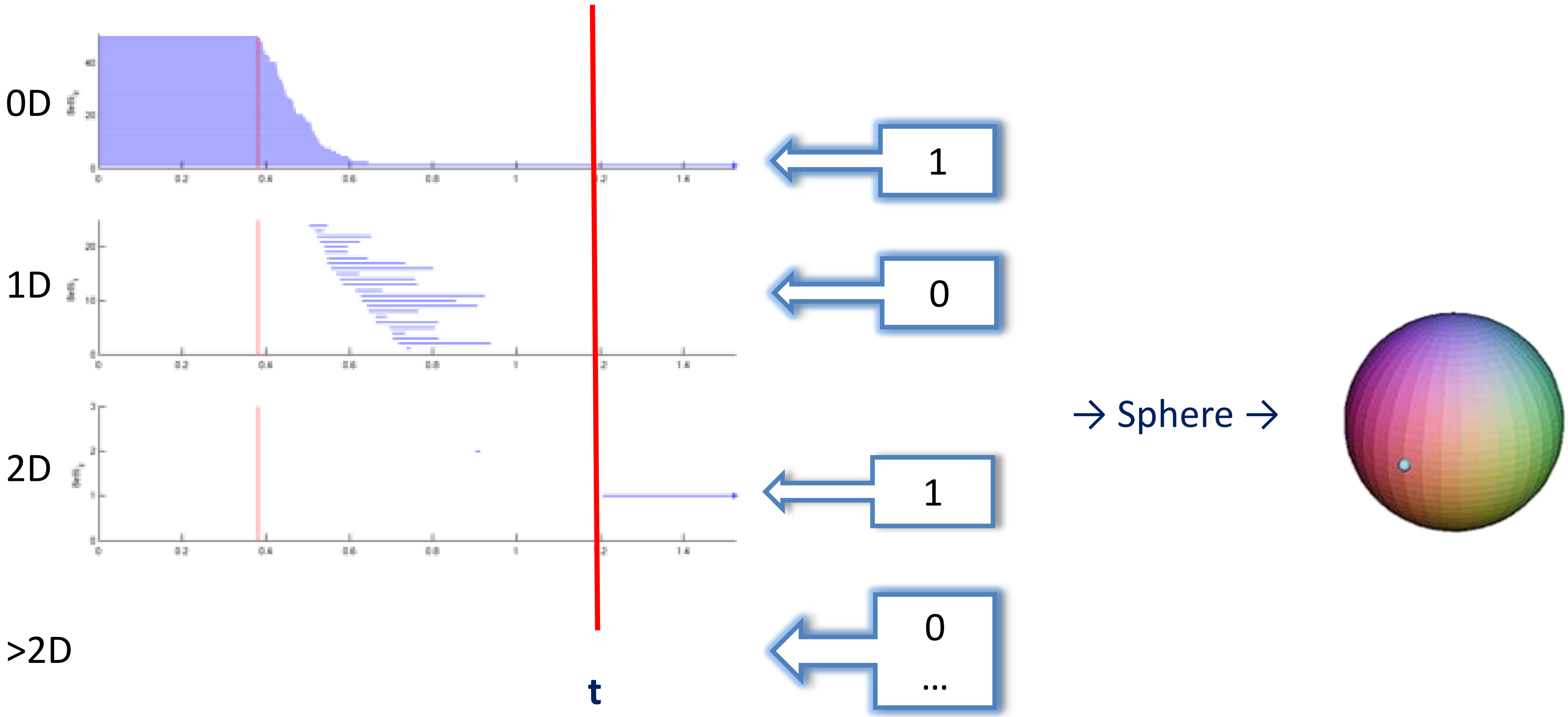
Example: Sphere





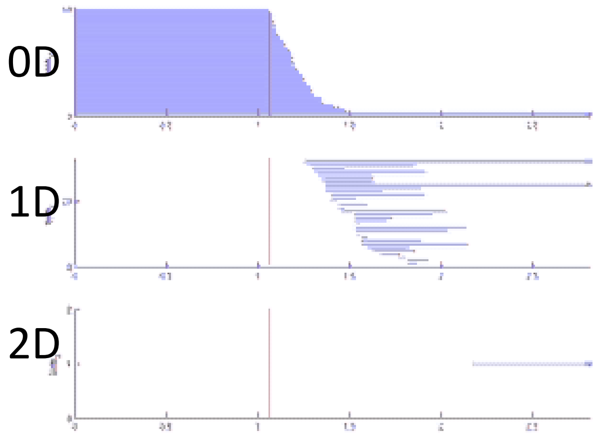


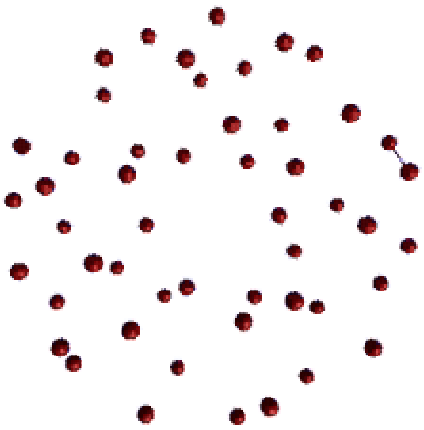
Topological barcode of a sphere

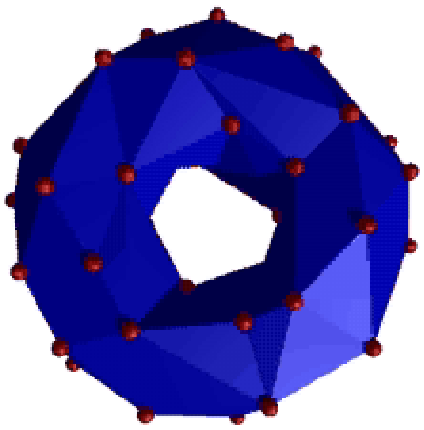


Topological barcode (1, 0 , 1, 0, 0 …)

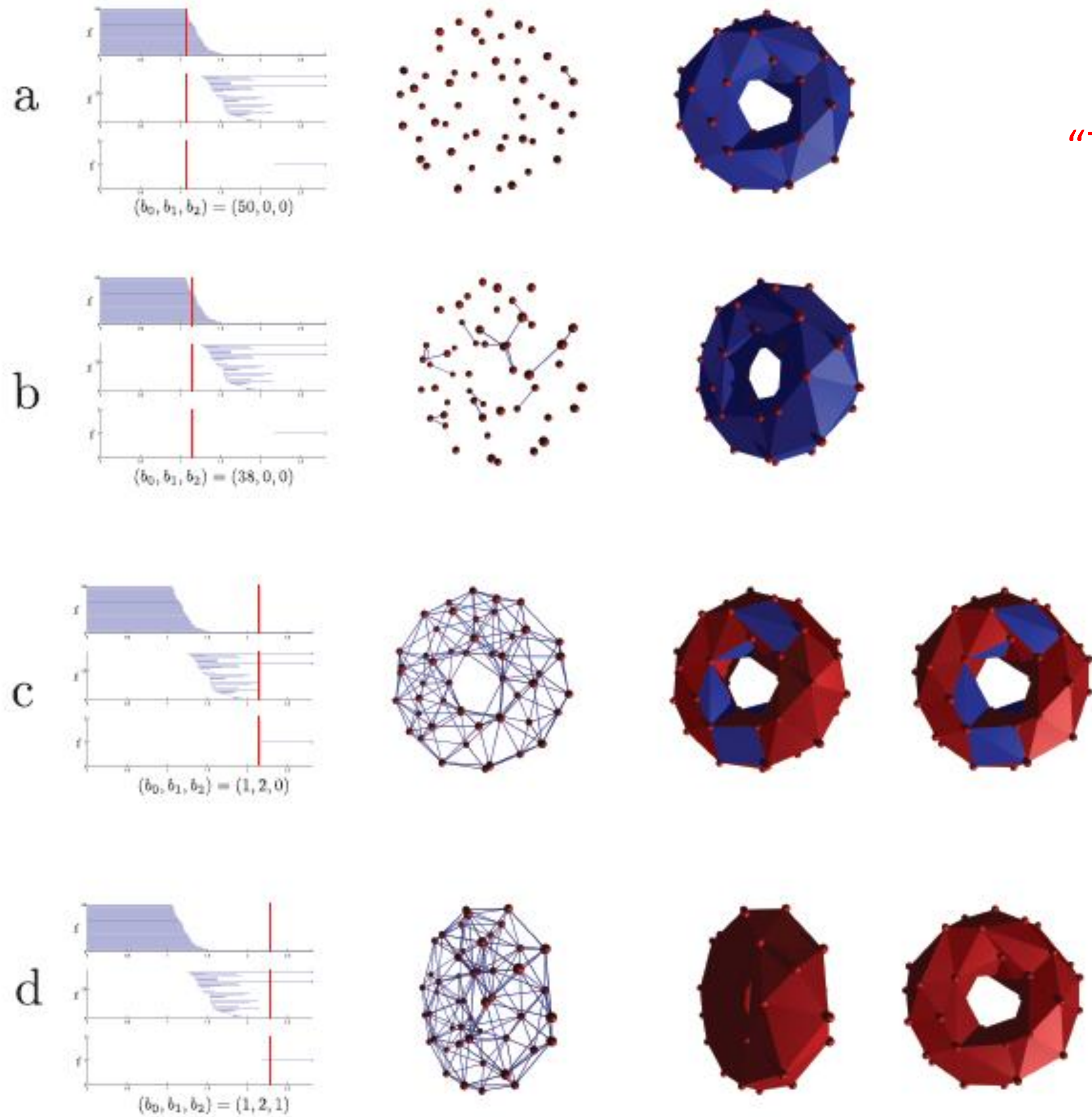
The unfolding of the topological information







The unfolding of the topological information



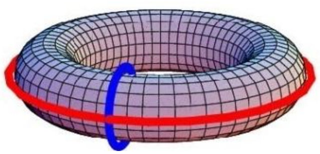
"Topological barcode"

(1, 2 , 1, 0,...)

↓

Torus

↓



“Topology! The stratosphere of human thought! In the twenty-fourth century it might possibly be of use to someone...”

A. Solzhenitsyn, “*The First Circle*” (1955—58)

Homology: An Idea Whose Time Has Come

B. Cipra, *SIAM News, Vol. 42(10),* (2009)

Summary

1. Simplexes and simplicial complexes
2. Boundaries and orientations
3. Homologous cycles
4. Homological group

Next: Neuroscience applications…

jPlex, computational topology software, Stanford University
http://comptop.stanford.edu/u/programs/jplex/index.html